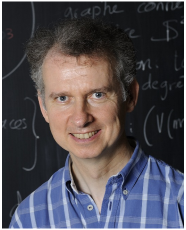
Lecture 2

3. GALTON-WATSON TREES

Key reference:

Jean-François Le Gall, **Random trees and applications**, *Probability Surveys* **2** (2005) pp.245-311.



Ordered trees

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This is not too much of a restriction if what we're really interested in is labelled unordered trees, since it's always possible to obtain a rooted ordered tree from a labelled one: for example, root at the vertex labelled 1 and order the children of a vertex from left to right in increasing order of label.

Ordered trees: some notation

We will use the Ulam-Harris labelling. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n,$$

where $\mathbb{N}^0 = \{\emptyset\}$.

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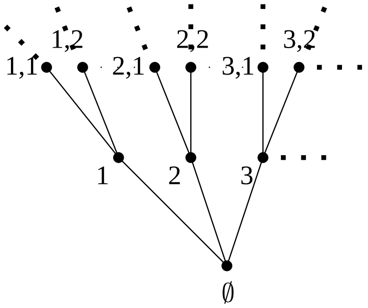
$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n,$$

where $\mathbb{N}^0 = \{\emptyset\}$. An element $u \in \mathcal{U}$ is a sequence $u = (u^1, u^2, \ldots, u^n)$ of natural numbers representing a point in an infinitary tree:



Thus the label of a vertex indicates its genealogy.

Ordered trees: some notation

Write $|u| = n$ for the generation of u.

u has parent $p(u) = (u^1, u^2, \ldots, u^{n-1})$.

u has children $u1, u2, \ldots$.

We root the tree at $\emptyset$.

Ordered trees

A (finite) rooted, ordered tree $\mathbf{t}$ is a finite subset of $\mathcal{U}$ such that

- $\emptyset \in \mathbf{t}$
- for all $u \in \mathbf{t}$ such that $u \neq \emptyset$, $p(u) \in \mathbf{t}$
- for all $u \in \mathbf{t}$, there exists $c(u) \in \mathbb{Z}_+$ such that for $j \in \mathbb{N}$, $uj \in \mathbf{t}$ iff $1 \leq j \leq c(u)$.

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Write $\#(\mathbf{t})$ for the size (number of vertices) of $\mathbf{t}$ and note that

$$\#(\mathbf{t}) = 1 + \sum_{u \in \mathbf{t}} c(u).$$

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Write $\mathbf{T}$ for the set of all rooted ordered trees.

Two ways of encoding a tree

Consider a rooted ordered tree $\mathbf{t} \in \mathbf{T}$.

It will be convenient to encode this tree in terms of discrete functions which are easier to manipulate.

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We will do this is two different ways:

- the height function
- the depth-first walk.

Height function

Suppose that $\mathbf{t}$ has n vertices. Let them be $v_0, v_1, \ldots, v_{n-1}$, listed in lexicographical order.

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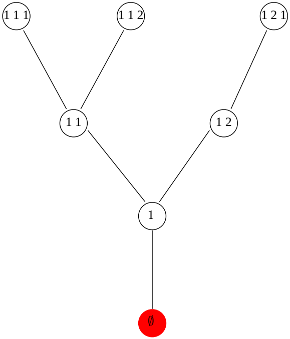
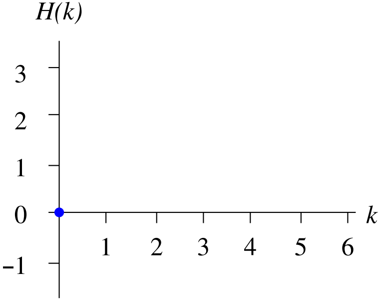
Then the height function is defined by

$$H(k) = |v_k|, \quad 0 \leq k \leq n-1.$$

Height function





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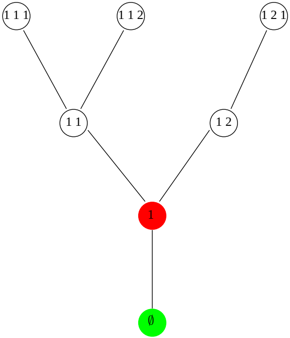

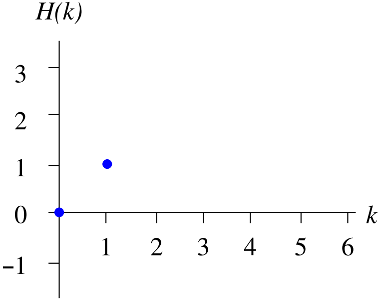

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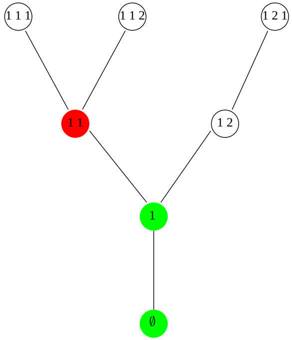

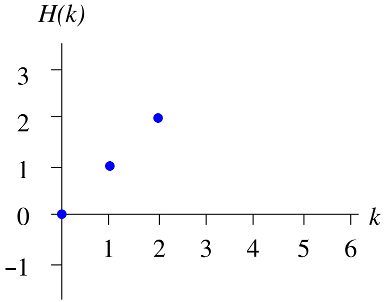

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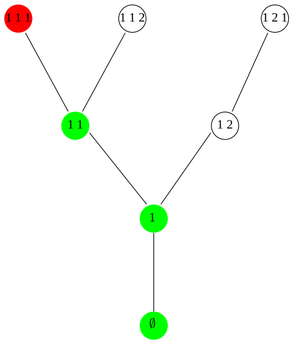

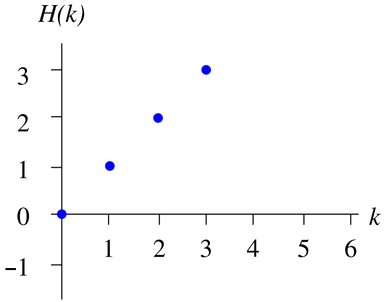

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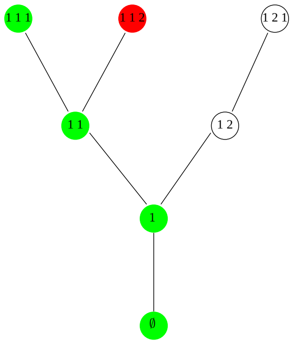

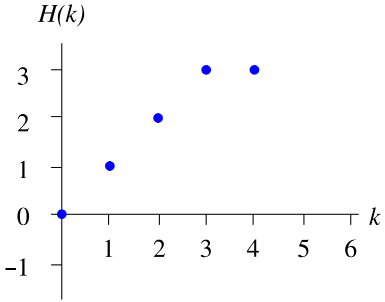

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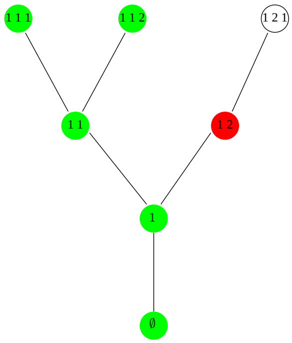

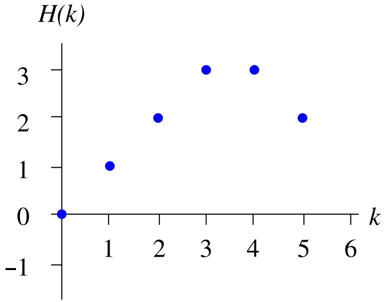

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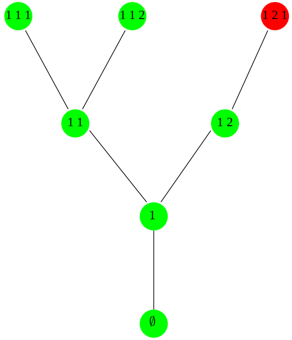

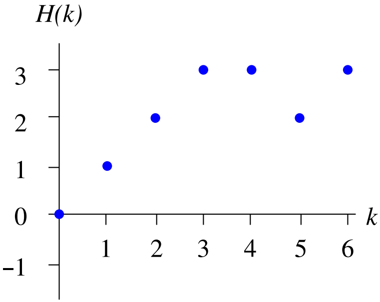

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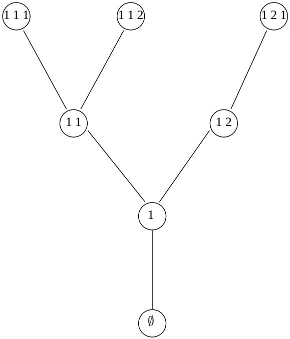

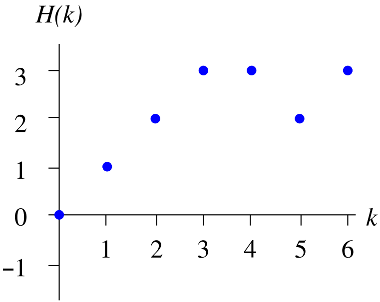

We can recover the tree from its height function.

Depth-first walk

Recall that $c(v)$ is the number of children of v, and that $v_0, v_1, \ldots, v_{n-1}$ is a list of the vertices of $\mathbf{t}$ in lexicographical order.

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$$X(0) = 0,$$
$$X(i) = \sum_{j=0}^{i-1} (c(v_j) - 1), \text{ for } 1 \leq i \leq n.$$

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In other words,

$$X(i+1) = X(i) + c(v_i) - 1, \quad 0 \leq i \leq n-1.$$

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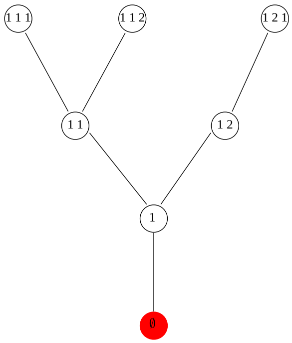

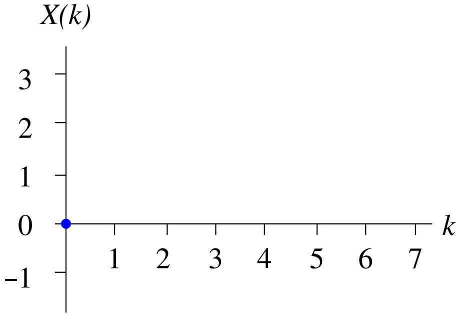

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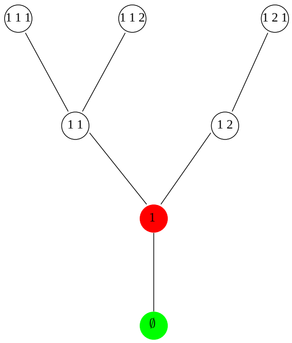

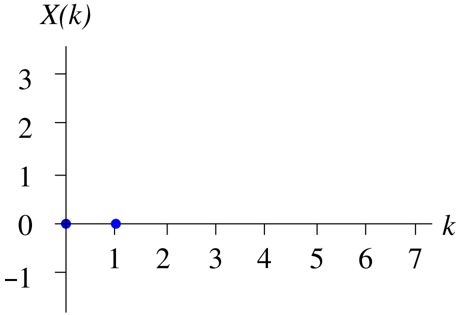

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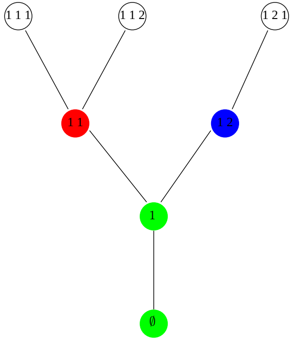

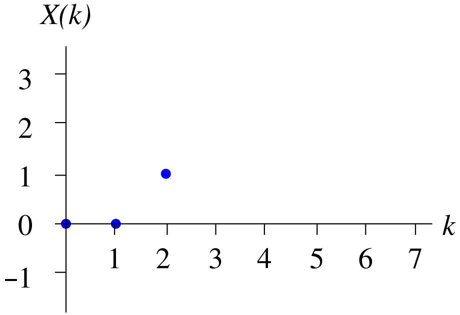

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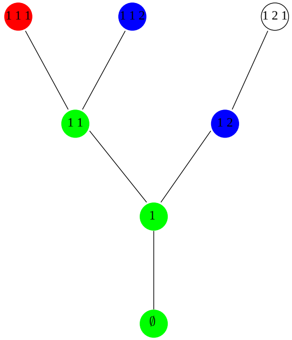

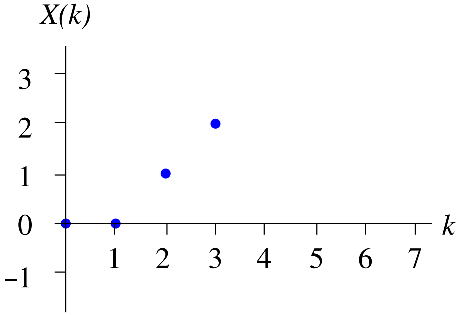

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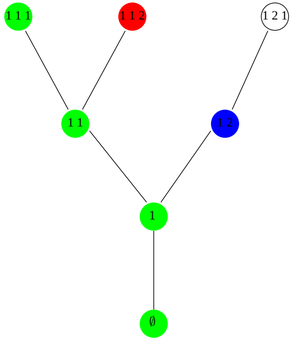

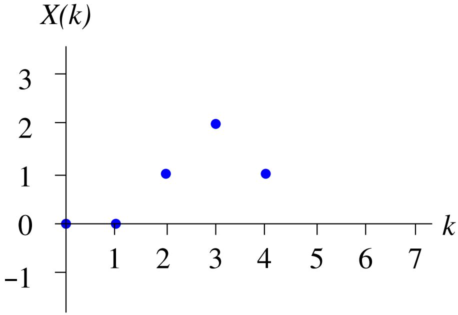

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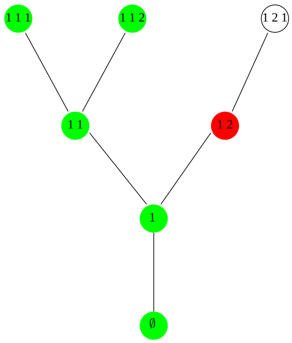

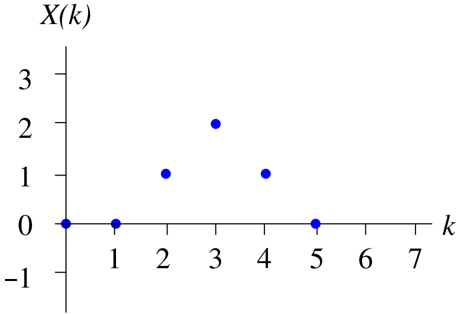

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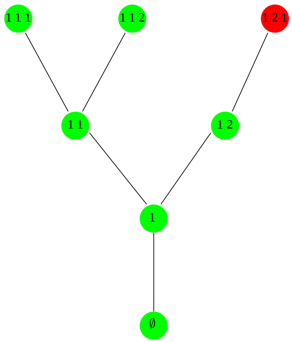

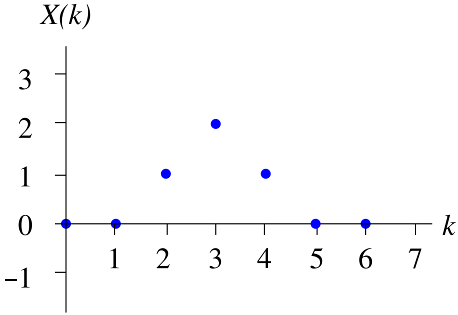

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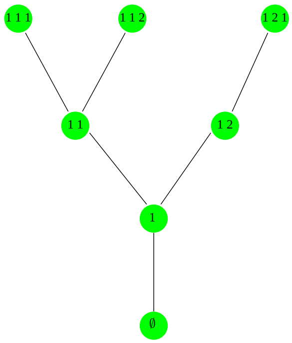

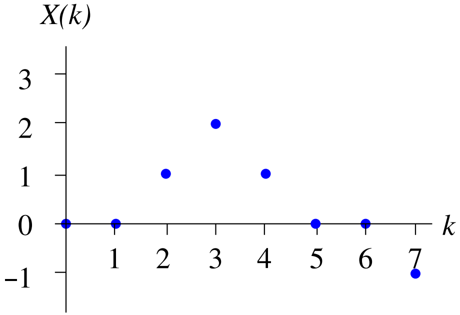

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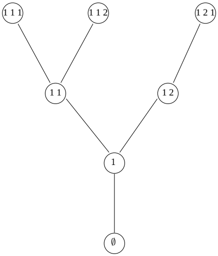

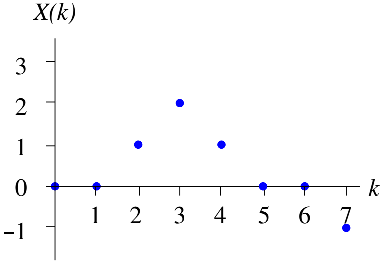

It is less easy to see that the depth-first walk also encodes the tree.

Proposition

For $0 \leq i \leq n-1$,

$$H(i) = \#\left\{0 \leq j \leq i-1 : X(j) = \min_{j \leq k \leq i} X(k)\right\}.$$

Random discrete trees

From a probabilistic perspective, a natural probability measure on trees is that generated by a so-called Galton-Watson branching process. We will see in a moment that this is a good thing to do from a combinatorial perspective too!

Galton-Watson processes

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- Each child reproduces as an independent copy of the original individual.

Z_n gives the number of individuals in generation n (in particular, $Z_0 = 1$). The process $(Z_n)_{n \geq 0}$ is a Markov chain with an absorbing state at 0.

Galton-Watson processes

In order to avoid special cases, we will assume that $p(0) > 0$ and $p(0) + p(1) < 1$. This means that it's always possible for the branching process to die out and we won't have every individual that gives birth just deterministically having a single offspring.

Generating functions

Probability generating functions play a key role in the analysis of branching processes. Let

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Then if $G_n(s) = \mathbb{E}\left[s^{Z_n}\right]$, we get $G_1(s) = G(s)$ and, for $n \geq 2$,

$$G_n(s) = G_{n-1}(G(s)) = \underbrace{G(G(\ldots G}_{n \text{ times}}(s))) = G(G_{n-1}(s)).$$

Extinction probability

Let $q = \mathbb{P}\left(\text{population dies out}\right) = \mathbb{P}\left(\cup_{n=1}^{\infty}\{Z_n = 0\}\right)$. Since these events are nested ($Z_n = 0$ implies that $Z_{n+1} = 0$), we have

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Recall that each of the individuals in the first generation behaves exactly like the parent. We can think of each of them starting its own family, which is an independent copy of the original family. Moreover, the whole population dies out if and only if all of the subpopulations die out. If there are k individuals in the first generation, this occurs with probability q^k. So

$$q = \sum_{k=0}^{\infty} p(k) q^k = G(q).$$

Extinction probability

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Theorem

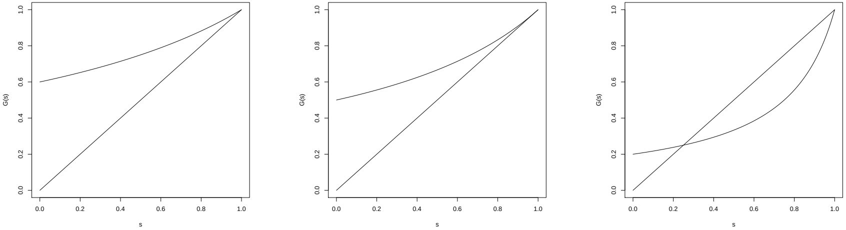
Suppose that $p(0) > 0$ and $p(0) + p(1) < 1$.

(a) *The equation $s = G(s)$ has at most two solutions in $[0, 1]$. The extinction probability q is the smallest non-negative root of the equation.*

(b) *Suppose that the offspring distribution has mean μ. Then*
   - *if $\mu \leq 1$ then $q = 1$;*
   - *if $\mu > 1$ then $q < 1$.*

Proof by picture

Solving $s = G(s)$:



$\mu < 1$ (subcritical) $\mu = 1$ (critical) $\mu > 1$ (supercritical)

Note that $\mathbb{P}(Z_n = 0) = G_n(0)$ and so $q = \lim_{n\to\infty} G_n(0)$.

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As before, call its depth-first walk X. Because the numbers of children of different individuals are i.i.d. X has a particularly nice form.

The depth-first walk of a Galton-Watson tree is a stopped random walk

Proposition

Let $(R(k), k \geq 0)$ *be a random walk with initial value 0 and step distribution* $\nu(k) = p(k+1), k \geq -1$. *Set*

$$M = \inf\{k \geq 0 : R(k) = -1\}.$$

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Now suppose that T *is a Galton-Watson tree with offspring distribution* p *and total progeny* N*. Then,*

$$(X(k), 0 \leq k \leq N) \stackrel{d}{=} (R(k), 0 \leq k \leq M).$$

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[Careful proof: see Le Gall (2005).]

Critical Galton-Watson trees

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The critical case turns out to be the right one to consider in order to capture various natural combinatorial models.

Uniform random trees revisited

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Other combinatorial trees (in disguise)

Let T be a Galton-Watson tree with offspring distribution p and total progeny N.

Exercise

1. *If $p(k) = 2^{-k-1}$, $k \geq 0$ (i.e. Geometric(1/2) offspring distribution) then conditional on $N = n$, the tree is uniform on the set of ordered trees with n vertices.*
2. *If $p(0) = 1/2$ and $p(2) = 1/2$ then, conditional on $N = n$ (for n odd), the tree is uniform on the set of (complete) binary trees.*

Galton-Watson trees conditioned on their total progeny: finite variance case

Suppose now that we have offspring variance $\sigma^2 := \sum_{k=1}^{\infty}(k-1)^2 p(k) \in (0, \infty)$ (which is the case for all the examples we have seen so far).

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Standing assumption: $\mathbb{P}(N = n) > 0$ for all n sufficiently large. (This is true if, for example, $p(1) > 0$.)

Galton-Watson trees conditioned on their total progeny: finite variance case

Write $(X^n(k), 0 \leq k \leq n)$ for the depth-first walk conditioned on $\{N = n\}$. Then there is a conditional version of Donsker's theorem:

Theorem (Kaigh (1976))

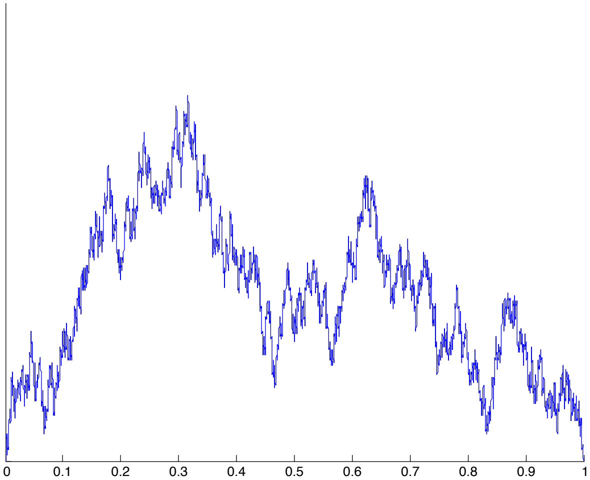
As $n \to \infty$,

$$\frac{1}{\sigma\sqrt{n}}(X^n(\lfloor nt \rfloor), 0 \leq t \leq 1) \xrightarrow{d} (e(t), 0 \leq t \leq 1),$$

where $(e(t), 0 \leq t \leq 1)$ is a standard Brownian excursion.

[W.D. Kaigh, **An invariance principle for random walk conditioned by a late return to zero**, *Annals of Probability* **4** (1976) pp.115-121.]

Brownian excursion



[Picture by Igor Kortchemski]

Brownian excursion

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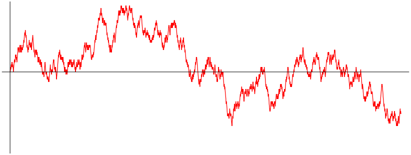
For example, let W be a standard Brownian motion.



Fix $s > 0$. Let

$$g_s = \sup\{t \leq s : W(t) = 0\} \text{ and } d_s = \inf\{t \geq s : W(t) = 0\}.$$

Note that $W(s) \neq 0$ with probability 1, so that $\mathbb{P}(g_s < s < d_s) = 1$. Then for $t \in [0, 1]$ define

$$e(t) = \frac{|W(g_s + t(d_s - g_s))|}{\sqrt{d_s - g_s}}.$$

It turns out that the distribution of $(e(t), 0 \leq t \leq 1)$ is independent of s.

Convergence of the coding processes

Let $(H^n(i), 0 \leq i \leq n)$ be the height process of a critical Galton-Watson tree with offspring variance $\sigma^2 \in (0, \infty)$, conditioned to have total progeny n. (Since the tree is random, we refer to the height process rather than function.)

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Theorem

As $n \to \infty$,

$$\frac{\sigma}{\sqrt{n}} \left(H^n(\lfloor nt \rfloor), 0 \leq t \leq 1\right) \stackrel{d}{\to} 2\left(e(t), 0 \leq t \leq 1\right),$$

where $(e(t), 0 \leq t \leq 1)$ *is a standard Brownian excursion.*

Actually, I'm going to cheat...

Consider the unconditioned random walk $(X(k), k \geq 0)$ (without stopping). This is the depth-first walk of a sequence of i.i.d. unconditioned Galton-Watson trees: the random walk X begins encoding a new tree every time it attains a new minimum. It is technically easier to work without the conditioning.

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Define the height process H for all $i \geq 0$ via $H(0) = 0$ and, for $i \geq 1$,

$$H(i) = \# \left\{ 0 \leq j \leq i-1 : X(j) = \min_{j \leq k \leq i} X(k) \right\}.$$

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We have

$$\frac{1}{\sigma \sqrt{n}} (X(\lfloor nt \rfloor), t \geq 0) \xrightarrow{d} (W(t), t \geq 0)$$

as $n \to \infty$.

An unconditioned result

Proposition

As $n \to \infty$*,*

$$\frac{\sigma}{\sqrt{n}}(H(\lfloor nt \rfloor), t \geq 0) \to 2\left(W(t) - \min_{0 \leq s \leq t} W(s), t \geq 0\right)$$

in the sense of finite-dimensional distributions, i.e. if $0 \leq t_1 \leq t_2 \leq \cdots \leq t_m$ *then*

$$\frac{\sigma}{\sqrt{n}}(H(\lfloor nt_1 \rfloor), \ldots, H(\lfloor nt_m \rfloor))$$
$$\xrightarrow{d} 2\left(W(t_1) - \min_{0 \leq s \leq t_1} W(s), \ldots, W(t_m) - \min_{0 \leq s \leq t_m} W(s)\right).$$

[Approach due to Marckert & Mokkadem, **The depth first processes of Galton-Watson trees converge to the same Brownian excursion**, *Annals of Probability* **31** (2003), pp.1655-1678]