# Lecture 1: Brownian motion and stochastic integrals

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Escuela de Probabilidad
CIMAT, Guanajuato, Septiembre 2016

# Outline

1. Lecture 1 : Brownian motion, martingales and stochastic integrals.
2. Lecture 2 : Introduction to Malliavin calculus.
3. Lecture 3 : Stein's method for normal approximations.
4. Lecture 4 : Applications to functionals of the fractional Brownian motion.

# Multivariate normal distribution

- A random vector $X = (X_1, \dots, X_n)$ has the *multivariate normal distribution* $N(\mu, \Sigma)$, if its characteristic function is

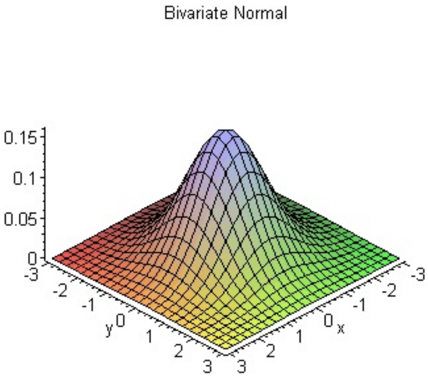
$$E\left(e^{i\langle u, X\rangle}\right) = \exp\left(i\langle u, \mu\rangle - \frac{1}{2}u^T \Sigma u\right), \; u \in \mathbb{R}^n,$$

  where $\mu \in \mathbb{R}^n$ and $\Sigma$ is an $n \times n$ symmetric and nonnegative definite matrix.

- $\mu = (E(X_1), \dots, E(X_n))$
- $\Sigma_{ij} = \mathrm{Cov}(X_i, X_j)$
- If $X$ has the $N(\mu, \Sigma)$ distribution, then $Y = AX + b$, where $A$ is an $m \times n$ matrix and $b \in \mathbb{R}^m$, has the $N(A\mu + b, A\Sigma A^T)$ distribution.

- If $\Sigma$ is nonsingular, then $X$ has a density given by

$$f(x) = (2\pi)^{-\frac{n}{2}} (\det \Sigma)^{-\frac{1}{2}} \exp\left( -\frac{1}{2} (x-\mu)^T \Sigma^{-1} (x-\mu) \right).$$



# Stochastic processes

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- The probabilities on $\mathbb{R}^n$, $n \geq 1$,

$$P_{t_1,\dots,t_n} = P \circ (X_{t_1}, \dots, X_{t_n})^{-1}$$

  where $0 \leq t_1 < \cdots < t_n$, are called the *finite-dimensional marginal distributions* of the process $X$.

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where $0 \leq t_1 < \cdots < t_n$, are called the *finite-dimensional marginal distributions* of the process $X$.
- For every $\omega \in \Omega$, the mapping
$$t \to X_t(\omega)$$
is called a *trajectory* of the process $X$.

## Theorem (Kolmogorov's extension theorem)

*Consider a family of probability measures*

$$\{P_{t_1,\dots,t_n},\ 0 \le t_1 < \cdots < t_n, n \ge 1\}$$

*such that :*

(i) $P_{t_1,\dots,t_n}$ *is a probability on* $\mathbb{R}^n$.

(ii) *(Consistency condition) : If* $\{t_{k_1} < \cdots < t_{k_m}\} \subset \{t_1 < \cdots < t_n\}$, *then* $P_{t_{k_1},\dots,t_{k_m}}$ *is the marginal of* $P_{t_1,\dots,t_n}$, *corresponding to the indexes* $k_1, \dots, k_m$.

*Then, there exists a stochastic process* $\{X_t, t \ge 0\}$ *defined in some probability space* $(\Omega, \mathcal{F}, P)$, *which has the family* $\{P_{t_1,\dots,t_n}\}$ *as finite-dimensional marginal distributions.*

- Take $\Omega$ as the set of all functions $\omega : [0, \infty) \to \mathbb{R}$, $\mathcal{F}$ the $\sigma$-algebra generated by cylindrical sets, extend the probability from cylindrical sets to $\mathcal{F}$, and set $X_t(\omega) = \omega(t)$.

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- Suppose $\mu : \mathbb{R}_+ \to \mathbb{R}$, and $\Gamma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is symmetric and nonnegative definite :

$$\sum_{i,j=1}^{n} \Gamma(t_i, t_j) a_i a_j \geq 0, \quad \forall\, t_i \geq 0,\ a_i \in \mathbb{R}.$$

Then there exists a Gaussian process with mean $\mu$ and covariance function $\Gamma$.

# Equivalent processes

- Two processes, $X$, $Y$ are *equivalent* (or $X$ is a version of $Y$) if for all $t \geq 0$,
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- Two equivalent processes may have quite different trajectories. For example, the processes $X_t = 0$ for all $t \geq 0$ and
$$Y_t = \begin{cases} 0 & \text{if} \quad \xi \neq t \\ 1 & \text{if} \quad \xi = t \end{cases}$$
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- Two processes $X$ and $Y$ are said to be *indistinguishable* if
$$X_t(\omega) = Y_t(\omega)$$
for all $t \geq 0$ and for all $\omega \in \Omega^*$, with $P(\Omega^*) = 1$.
- Two equivalent processes with right-continuous trajectories are indistinguishable.

# Regularity of trajectories

**Theorem (Kolmogorov's continuity theorem)**

*Suppose that $X = \{X_t, t \in [0, T]\}$ satisfies*

$$E(|X_t - X_s|^\beta) \le K|t - s|^{1+\alpha},$$

*for all $s, t \in [0, T]$, and for some constants $\beta, \alpha > 0$. Then, there exists a version $\widetilde{X}$ of $X$ such that, if $\gamma < \alpha/\beta$,*

$$|\widetilde{X}_t - \widetilde{X}_s| \le G_\gamma |t - s|^\gamma$$

*for all $s, t \in [0, T]$, where $G_\gamma$ is a random variable.*

- The trajectories of $\widetilde{X}$ are Hölder continuous of order $\gamma$ for any $\gamma < \alpha/\beta$.

# Brownian motion

A stochastic process $B = \{B_t, t \geq 0\}$ is called a *Brownian motion* if :

i) $B_0 = 0$ almost surely.

ii) *Independent increments :* For all $0 \leq t_1 < \cdots < t_n$ the increments $B_{t_n} - B_{t_{n-1}}, \ldots, B_{t_2} - B_{t_1}$, are independent random variables.

iii) If $0 \leq s < t$, the increment $B_t - B_s$ has the normal distribution $N(0, t - s)$.

iv) With probability one, $t \to B_t(\omega)$ is continuous.

## Proposition

*Properties i), ii), iii) are equivalent to :*

($\star$) *B is a Gaussian process with mean zero and covariance*

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*Proof :*

a) Suppose i), i) and iii). The distribution of $(B_{t_1}, \dots, B_{t_n})$, for $0 < t_1 < \cdots < t_n$, is normal, because this vector is a linear transformation of $\left(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}\right)$ which has independent and normal components.

The mean is zero, and for $s < t$, the covariance is

$$E(B_s B_t) = E(B_s(B_t - B_s + B_s)) = E(B_s(B_t - B_s)) + E(B_s^2) = s.$$

b) The converse is also easy to show. $\square$

# Construction of the Brownian motion

**1.** The function $\Gamma(s,t) = \min(s,t)$ is symmetric and nonnegative definite because it can be written as

$$\min(s,t) = \int_0^\infty \mathbf{1}_{[0,s]}(r)\mathbf{1}_{[0,t]}(r)dr,$$

so

$$\sum_{i,j=1}^n a_i a_j \min(t_i,t_j) = \sum_{i,j=1}^n a_i a_j \int_0^\infty \mathbf{1}_{[0,t_i]}(r)\mathbf{1}_{[0,t_j]}(r)dr = \int_0^\infty \left[\sum_{i=1}^n a_i \mathbf{1}_{[0,t_i]}(r)\right]^2 dr \geq 0.$$

Therefore, by Kolmogorov's extension theorem there exists a Gaussian process $B$ with zero mean and covariance function $\min(s,t)$.

2. The process $B$ satisfies

$$E\left[(B_t - B_s)^{2k}\right] = \frac{(2k)!}{2^k k!}(t-s)^k, \quad s \le t$$

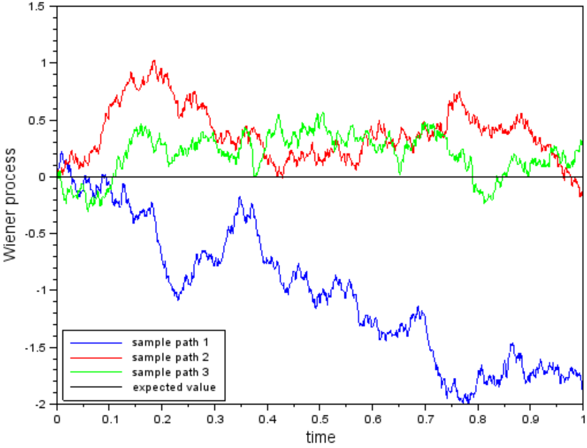
for any $k \geq 1$, because the distribution of $B_t - B_s$ is $N(0, t-s)$.

3. Therefore, by the Kolmogorov's continuity theorem, there exist a version $\widetilde{B}$ of $B$, such that $\widetilde{B}$ has Hölder continuous trajectories of order $\gamma$ for any $\gamma < \frac{k-1}{2k}$ on any interval $[0, T]$. This implies that the paths are $\gamma$-Hölder on $[0, T]$ for any $\gamma < \frac{1}{2}$ and for any $T > 0$.

# Brownian motion and random walk

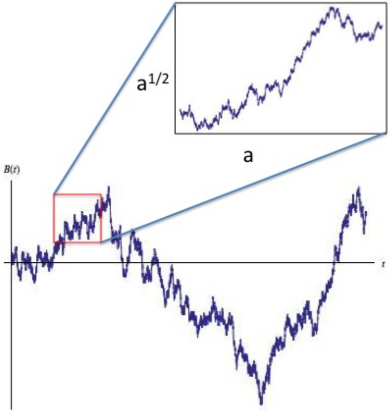
- Let $\{\xi_k, 1 \leq k \leq n\}$ be independent and identically distributed random variables with zero mean and variance one.
- Define $S_n(0) = 0$,
$$S_n(\frac{kT}{n}) = \sqrt{T}\frac{\xi_1 + \cdots + \xi_k}{\sqrt{n}}, \quad k = 1, \ldots, n$$
and extend $S_n(t)$ to $t \in [0, T]$ by linear interpolation.
- *Donsker Invariance Principle* : The law of the random walk $S_n$ on $C([0, T])$ converges to the *Wiener measure*, which is the law of the Brownian motion. That is, that for any continuous and bounded function $\varphi : C([0, T]) \to \mathbb{R}$,
$$E(\varphi(S_n)) \stackrel{n\to\infty}{\longrightarrow} E(\varphi(B)),$$

# Simulations of Brownian motion



# Basic properties

1. *Selfsimilarity :*
   For any $a > 0$, the process $\{a^{-\frac{1}{2}} B_{at}, t \geq 0\}$ is also a Brownian motion.



2. For any $h > 0$, the process $\{B_{t+h} - B_h, t \geq 0\}$ is a Brownian motion.
3. The process $\{-B_t, t \geq 0\}$ is a Brownian motion.
4. The process

$$X_t = \begin{cases} tB_{1/t}, & t > 0 \\ 0, & t = 0 \end{cases}$$

is a Brownian motion.

# Quadratic variation

Fix a time interval $[0, t]$ and consider a partition

$$\pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}.$$

Define $\Delta t_k = t_k - t_{k-1}$, $\Delta B_k = B_{t_k} - B_{t_{k-1}}$ and $|\pi| = \max_{1\leq k\leq n} \Delta t_k$.

**Proposition**

*The following convergence holds in $L^2$ :*

$$\lim_{|\pi|\to 0} \sum_{k=1}^{n} (\Delta B_k)^2 = t.$$

- We can say that $(\Delta B_t)^2 \sim \Delta t$

*Proof :* Set $\xi_k = (\Delta B_k)^2 - \Delta t_k$. The random variables $\xi_k$ are independent and centered. Thus,

$$\begin{aligned}
E\left[\left(\sum_{k=1}^{n} (\Delta B_k)^2 - t\right)^2\right] &= E\left[\left(\sum_{k=1}^{n} \xi_k\right)^2\right] = \sum_{k=1}^{n} E\left[\xi_k^2\right] \\
&= \sum_{k=1}^{n} \left[3\,(\Delta t_k)^2 - 2\,(\Delta t_k)^2 + (\Delta t_k)^2\right] \\
&= 2\sum_{k=1}^{n} (\Delta t_k)^2 \le 2t|\pi| \overset{|\pi|\to 0}{\longrightarrow} 0. \quad \square
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$$

*Exercise* : Using the Borel-Cantelli lemma, show that if $\{\pi^n\}$ is a sequence of partitions of $[0, t]$ such that $\sum_n |\pi^n| < \infty$, then $\sum_{k=1}^{n}\left(\Delta B_k\right)^2$ converges almost surely to $t$.

# Infinite total variation

- Define

$$V_t = \sup_{\pi} \sum_{k=1}^{n} |\Delta B_k|$$

- Then,

$$P(V_t = \infty) = 1.$$

In fact, using the continuity of the trajectories of the Brownian motion, we have, on the set $V < \infty$,

$$\sum_{k=1}^{n} (\Delta B_k)^2 \leq \sup_k |\Delta B_k| \left( \sum_{k=1}^{n} |\Delta B_k| \right) \leq V \sup_k |\Delta B_k| \stackrel{|\pi| \to 0}{\longrightarrow} 0.$$

Then, $V < \infty$ contradicts the fact that $\sum_{k=1}^{n} (\Delta B_k)^2$ converges in $L^2$ to $t$ as $|\pi| \to 0$.

# Martingales

- We assume that $\{\mathcal{F}_t, t \geq 0\}$ is an increasing family of $\sigma$-fields, contained in $\mathcal{F}$ (*filtration*).

## Definition

An adapted process $M = \{M_t, t \geq 0\}$ is called a *martingale* with respect to $\mathcal{F}_t$ if

(i) For all $t \geq 0$, $E(|M_t|) < \infty$.

(ii) For each $s \leq t$, $E(M_t|\mathcal{F}_s) = M_s$.

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- Property (ii) can also be written as :

$$E(M_t - M_s|\mathcal{F}_s) = 0.$$

# *Examples :*

Let $B_t$ be a Brownian motion and let $\mathcal{F}_t$ be the filtration generated by $B_t$ :

$$\mathcal{F}_t = \sigma\{B_s, 0 \leq s \leq t\}.$$

Then, the processes

$$M_t^{(1)} = B_t$$
$$M_t^{(2)} = B_t^2 - t$$
$$M_t^{(3)} = \exp(aB_t - \frac{a^2 t}{2})$$

where $a \in \mathbb{R}$, are martingales.

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2. For $B_t^2 - t$, we can write, using the properties of the conditional expectation, for $s < t$

$$\begin{aligned} E(B_t^2|\mathcal{F}_s) &= E((B_t - B_s + B_s)^2 |\mathcal{F}_s) \\ &= E((B_t - B_s)^2 |\mathcal{F}_s) + 2E((B_t - B_s) B_s|\mathcal{F}_s) \\ &\quad + E(B_s^2|\mathcal{F}_s) \\ &= E(B_t - B_s)^2 + 2B_s E((B_t - B_s)|\mathcal{F}_s) + B_s^2 \\ &= t - s + B_s^2. \end{aligned}$$

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3. Finally, for $\exp(aB_t - \frac{a^2t}{2})$ we have

$$\begin{aligned} E(e^{aB_t - \frac{a^2t}{2}}|\mathcal{F}_s) &= e^{aB_s}E(e^{a(B_t - B_s) - \frac{a^2t}{2}}|\mathcal{F}_s) \\ &= e^{aB_s}E(e^{a(B_t - B_s) - \frac{a^2t}{2}}) \\ &= e^{aB_s}e^{\frac{a^2(t-s)}{2} - \frac{a^2t}{2}} = e^{aB_s - \frac{a^2s}{2}}. \end{aligned}$$

# Doob's maximal inequalities

## Theorem

*Let* $\{M_t, t \in [0, T]\}$ *be a continuous martingale such that* $E(|M_T|^p) < \infty$ *for some* $p \geq 1$. *Then, for all* $\lambda > 0$ *we have*

$$P\left(\sup_{0 \leq t \leq T} |M_t| > \lambda\right) \leq \frac{1}{\lambda^p} E(|M_T|^p). \tag{1}$$

*If* $p > 1$, *then*

$$E\left(\sup_{0 \leq t \leq T} |M_t|^p\right) \leq \left(\frac{p}{p-1}\right)^p E(|M_T|^p). \tag{2}$$

# The Wiener integral

- The integral of a step function $\varphi_t = \sum_{j=0}^{m-1} a_j \mathbf{1}_{(t_j, t_{j+1}]}(t) \in \mathcal{E}$ with respect to a Brownian motion $B$ on $[0, T]$ is defined by

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- The mapping $\varphi \to \int_0^T \varphi_t dB_t$ from $\mathcal{E} \subset L^2([0, T])$ to $L^2(\Omega)$ is linear and isometric :

$$E\left[\left(\int_0^T \varphi_t dB_t\right)^2\right] = \sum_{j=0}^{m-1} a_j^2 (t_{j+1} - t_j) = \int_0^T \varphi_t^2 dt = \|\varphi\|_{L^2([0,T])}^2.$$

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- $\mathcal{E}$ is a dense subspace of $L^2([0,T])$. Therefore, the mapping

$$\varphi \to B(\varphi) =: \int_0^T \varphi_t dB_t$$

can be extended to a linear isometry between $L^2([0,T])$ and the Gaussian subspace of $L^2(\Omega)$ spanned by $\{B_t, t \in [0,T]\}$.

# Progressively measurable processes

Let $\mathcal{F}_t$ be the $\sigma$-field generated by the random variables $\{B_s, 0 \leq s \leq t\}$ and the sets of probability zero.

## Definition

We say that $u = \{u_t, t \in [0, T]\}$ is *progressively measurable* if for any $t \in [0, T]$, the restriction of $u$ to $\Omega \times [0, t]$ is $\mathcal{F}_t \times \mathcal{B}([0, t])$-measurable.

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- Let $\mathcal{P}$ be the $\sigma$-field of sets $A \subset \Omega \times [0, T]$ such that $\mathbf{1}_A$ is progressively measurable.
- We denote by $L^2_T(\mathcal{P})$ the Hilbert space $L^2(\Omega \times [0, T], \mathcal{P}, P \times \ell)$, where $\ell$ is the Lebesgue measure, equipped with the norm

$$\|u\|^2 = E\left(\int_0^T u_s^2 ds\right).$$

# Stochastic integrals

- $u = \{u_t, t \in [0, T]\}$ is a *simple process* if

$$u_t = \sum_{j=0}^{n-1} \phi_j \mathbf{1}_{(t_j, t_{j+1}]}(t),$$

where $0 \le t_0 \le t_1 \le \cdots \le t_n = T$ and $\phi_j$ are $\mathcal{F}_{t_j}$-measurable random variables such that $E(\phi_j^2) < \infty$.

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- We define the stochastic integral of $u$ as

$$I(u) := \int_0^T u_t dB_t = \sum_{j=0}^{n-1} \phi_j \left( B_{t_{j+1}} - B_{t_j} \right).$$

(i) *Linearity :*

$$\int_0^T (au_t + bv_t)\, dB_t = a \int_0^T u_t dB_t + b \int_0^T v_t dB_t.$$

(ii) *Zero mean :*

$$E\left(\int_0^T u_t dB_t\right) = 0.$$

In fact,

$$\begin{aligned} E\left(\int_0^T u_t dB_t\right) &= \sum_{j=0}^{n-1} E\left[\phi_j\left(B_{t_{j+1}} - B_{t_j}\right)\right] \\ &= \sum_{j=0}^{n-1} E[\phi_j]E[B_{t_{j+1}} - B_{t_j}] = 0. \end{aligned}$$

(iii) *Isometry property* :

$$E\left[\left(\int_0^T u_t dB_t\right)^2\right] = E\left(\int_0^T u_t^2 dt\right).$$

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*Proof :* Set $\Delta B_j = B_{t_{j+1}} - B_{t_j}$. Then

$$E\left(\phi_i \phi_j \Delta B_i \Delta B_j\right) = \begin{cases} 0 & \text{if} \quad i \neq j \\ E\left(\phi_j^2\right)(t_{j+1} - t_j) & \text{if} \quad i = j \end{cases}$$

because if $i < j$ the random variables $\phi_i \phi_j \Delta B_i$ and $\Delta B_j$ are independent and if $i = j$ the random variables $\phi_i^2$ and $(\Delta B_i)^2$ are independent. So, we obtain

$$\begin{aligned} E\left[\left(\int_0^T u_t dB_t\right)^2\right] &= \sum_{i,j=0}^{n-1} E\left(\phi_i \phi_j \Delta B_i \Delta B_j\right) = \sum_{i=0}^{n-1} E\left(\phi_i^2\right)(t_{i+1} - t_i) \\ &= E\left(\int_0^T u_t^2 dt\right). \qquad \square \end{aligned}$$

## Proposition

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*Proof :*
Use the approximating sequence

$$u_t^{(n)} = \sum_{j=1}^{n-1} \left( \frac{n}{T} \int_{t_{j-1}}^{t_j} u_s ds \right) \mathbf{1}_{(t_j, t_{j+1}]}(t),$$

where $t_j = \frac{jT}{n}$. $\square$

## Proposition

*The stochastic integral can be extended to a linear isometry :*

$$I : L^2_T(\mathcal{P}) \to L^2(\Omega).$$

*Proof :* This follows form the fact that $\mathcal{E}$ is dense in $L^2_T(\mathcal{P})$. $\square$.

- The stochastic integral has the following properties :

$$E\left[I(u)\right] = 0$$

and

$$E\left[I(u)I(v)\right] = E\left(\int_0^\infty u_s v_s ds\right).$$

# *Example*

$$\boxed{\int_0^T B_t dB_t = \frac{1}{2} B_T^2 - \frac{1}{2} T}$$

*Proof :* The process $B_t$ being continuous in mean square, we can choose as approximating sequence

$$u_t^{(n)} = \sum_{j=1}^{n} B_{t_{j-1}} \mathbf{1}_{(t_{j-1}, t_j]}(t),$$

where $t_j = \frac{jT}{n}$, and we obtain

$$\begin{aligned}
\int_0^T B_t dB_t &= \lim_{n\to\infty} \sum_{j=1}^{n} B_{t_{j-1}} \left(B_{t_j} - B_{t_{j-1}}\right) \\
&= \frac{1}{2} \lim_{n\to\infty} \sum_{j=1}^{n} \left(B_{t_j}^2 - B_{t_{j-1}}^2\right) - \frac{1}{2} \lim_{n\to\infty} \sum_{j=1}^{n} \left(B_{t_j} - B_{t_{j-1}}\right)^2 \\
&= \frac{1}{2} B_T^2 - \frac{1}{2} T.
\end{aligned}$$

# Indefinite stochastic integrals

For $u \in L^2_T(\mathcal{P})$, we define the stochastic process

$$\boxed{\int_0^t u_s dB_s := \int_0^T u_s \mathbf{1}_{[0,t]}(s) dB_s}, \quad t \in [0, T]$$

## Proposition

*Let $u \in L^2_T(\mathcal{P})$. The indefinite stochastic integral*

$$M_t = \int_0^t u_s dB_s$$

*is a square integrable martingale with respect to the filtration $\mathcal{F}_t$ and admits a continuous version.*

# Itô's formula

- Itô's stochastic integral does not follow the chain rule of classical calculus.
- *Example :*

$$\int_0^t B_s dB_s = \frac{1}{2} B_t^2 - \frac{t}{2},$$

whereas if $x_t$ is a differentiable function such that $x_0 = 0$,

$$\int_0^t x_s dx_s = \int_0^t x_s x_s' ds = \frac{1}{2} x_t^2.$$

- In differential form

$$d(B_t^2) = 2B_t dB_t + dt,$$

and $dt$ comes from $(dB_t)^2 \sim dt$ and the Taylor expansion up to the second order.

- The stochastic integral can be extended (using convergence in probability) to progressively measurable processes satisfying $\int_0^T u_s^2 ds < \infty$ a.s. Denote the class of those processes by $L^2_{T,loc}(\mathcal{P})$.
- Denote by $L^1_{T,loc}(\mathcal{P})$ the space of progressively measurable processes $v = \{v_t, t \in [0, T]\}$ such that for $\int_0^T |v_s|\, ds < \infty$ a.s.

## Theorem (Itô's formula)

*Suppose that*

$$X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds,$$

*where $u \in L^2_{T,loc}(\mathcal{P})$ and $v \in L^1_{T,loc}(\mathcal{P})$. Let $f \in C^{1,2}$. Then,*

$$\begin{aligned} Y_t &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s) u_s dB_s \\ &\quad + \int_0^t \frac{\partial f}{\partial x}(s, X_s) v_s ds + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s) u_s^2 ds. \end{aligned}$$

- In differential notation Itô's formula can be written as

$$df(t,X_t) = \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,X_s)\,(dX_t)^2\,,$$

where $(dX_t)^2$ is computed from

$$dX_t = u_t dB_t + v_t dt,$$

using the product rule

| $\times$ | $dB_t$ | $dt$ |
|---|---|---|
| $dB_t$ | $dt$ | 0 |
| $dt$ | 0 | 0 |

# Multiple stochastic integrals

- $L^2_s([0,T]^n)$ is the space of symmetric square integrable functions $f:[0,T]^n \to \mathbb{R}$.

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$$\|f\|^2_{L^2([0,T]^n)} = n! \int_{\Delta_n} f^2(t_1,\dots,t_n) dt_1 \cdots dt_n,$$

where

$$\Delta_n = \{(t_1,\dots,t_n) \in [0,T]^n : 0 < t_1 < \cdots < t_n < T\}.$$

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$$\Delta_n = \{(t_1, \dots, t_n) \in [0,T]^n : 0 < t_1 < \cdots < t_n < T\}.$$

- If $f : [0,T]^n \to \mathbb{R}$ we define its symmetrization as

$$\widetilde{f}(t_1, \dots, t_n) = \frac{1}{n!} \sum_{\sigma} f(t_{\sigma(1)}, \dots, t_{\sigma(n)}),$$

where the sum runs over all permutations $\sigma$ of $\{1, 2, \dots, n\}$.

- The *multiple stochastic integral* of $f \in L^2_s([0,T]^n)$ is defined as an iterated Itô integral :

$$I_n(f) = n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f(t_1, \dots, t_n) dB_{t_1} \cdots dB_{t_n}.$$

- The *multiple stochastic integral* of $f \in L_s^2([0,T]^n)$ is defined as an iterated Itô integral :

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- We have the following property :

$$E[I_n(f) I_m(g)] = \begin{cases} 0 & \text{if} \quad n \neq m \\ n! \langle f, g \rangle_{L^2([0,T]^n)} & \text{if} \quad n = m. \end{cases}$$

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- If $f \in L^2([0,T]^n)$ is not necessarily symmetric we define

$$I_n(f) = I_n(\widetilde{f}).$$

- The $n$th Hermite polynomial is defined by $h_0(x) = 1$ and

$$h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}), \quad n \geq 1.$$

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- Elementary properties :

$$\begin{aligned} h_n'(x) &= nh_{n-1}(x) \\ h_{n+1}(x) &= xh_n(x) - h_n'(x) = xh_n(x) - nh_{n-1}(x). \end{aligned}$$

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- For any $a \in \mathbb{R}$,

$$e^{az - \frac{1}{2}a^2} = \sum_{n=0}^{\infty} \frac{a^n}{n!} h_n(z).$$

### Theorem

*For any* $g \in L^2([0,T])$ *such that* $\|g\|_{L^2([0,T])} = 1$, *we have*

$$\boxed{I_n(g^{\otimes n}) = h_n\left(\int_0^T g_t dB_t\right)}$$

*where* $g^{\otimes n}(t_1, \ldots, t_n) = g(t_1) \cdots g(t_n)$.

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*where $g^{\otimes n}(t_1, \dots, t_n) = g(t_1)\cdots g(t_n)$.*

*Proof :*

(i) Fix $a \in \mathbb{R}$ and set

$$M_t = \exp\left(a\int_0^t g_s dB_s - \frac{1}{2}a^2\int_0^t g_s^2 ds\right).$$

One one hand, we have

$$M_T = e^{a\int_0^T g_s dB_s - \frac{1}{2}a^2} = \sum_{n=0}^{\infty} \frac{a^n}{n!} h_n\left(\int_0^T g_t dB_t\right).$$

(ii) On the other hand, using Itô's formula, we obtain

$$\begin{aligned} M_T &= 1 + \int_0^T a M_s g_s dB_s \\ &= 1 + aI_1(g) + a^2 \int_0^T g_s \int_0^s M_v g_v dB_v \\ &= 1 + aI_1(g) + a^2 \int_0^T g_s \int_0^s g_v dB_v + a^3 \int_0^T g_s \int_0^s M_v g_v dB_v \\ &= \sum_{n=0}^{\infty} \frac{a^n}{n!} I_n(g^{\otimes n}). \qquad \square \end{aligned}$$

# Product formula

Let $f \in L^2_s([0,T]^n)$, and $g \in L^2_s([0,T]^m)$. For any $r = 0, \dots, n \wedge m$, we define the *contraction* of $f$ and $g$ of order $r$ to be the element of $L^2([0,T]^{n+m-2r})$ defined by

$$\begin{aligned} & (f \otimes_r g)(t_1, \dots, t_{n-r}, s_1, \dots, s_{m-r}) \\ = & \int_{[0,T]^r} f(t_1, \dots, t_{n-r}, x_1, \dots, x_r) g(s_1, \dots, s_{m-r}, x_1, \dots, x_r) dx_1 \cdots dx_r. \end{aligned}$$

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- We denote by $f \widetilde{\otimes}_r g$ the symmetrization of $f \otimes_r g$.
- Product of two multiple stochastic integrals

$$\boxed{I_n(f) I_m(g) = \sum_{r=0}^{n \wedge m} r! \binom{n}{r} \binom{m}{r} I_{n+m-2r}(f \otimes_r g).}$$

# Wiener Chaos expansion

## Theorem

*$F \in L^2(\Omega)$ can be uniquely expanded into a sum of multiple stochastic integrals :*

$$F = E[F] + \sum_{n=1}^{\infty} I_n(f_n).$$

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- For any $n \geq 1$ we denote by $\mathcal{H}_n$ the closed subspace of $L^2(\Omega)$ formed by all multiple stochastic integrals of order $n$. For $n = 0$, $\mathcal{H}_0$ is the space of constants. Then, we have the orthogonal decomposition

$$L^2(\Omega) = \oplus_{n=0}^{\infty} \mathcal{H}_n.$$

- The theorem follows from the fact that if a random variable $G \in L^2(\Omega)$ is orthogonal to $\oplus_{n=0}^{\infty} \mathcal{H}_n$, then it is orthogonal to all random variables of the form $\left(\int_0^T g_t dW_t\right)^k$, where $g \in L^2([0, T])$, $k \geq 0$. This implies that $G$ is orthogonal to all the exponentials $\exp\left(\int_0^T g_t dW_t\right)$, which form a total set in $L^2(\Omega)$. So $G = 0$.

# Integral representation theorem

## Theorem

*Given* $F \in L^2(\Omega, \mathcal{F}_T, P)$ *there exists a unique process* $u$ *in the space* $L^2_T(\mathcal{P})$ *such that*

$$F = E[F] + \int_0^T u_t dB_t.$$

*Example :* $F = B_T^3$. By Itô's formula and integrating by parts

$$\begin{aligned} B_T^3 &= \int_0^T 3B_t^2 dB_t + 3\int_0^T B_t dt = \int_0^T 3B_t^2 dB_t + 3\left( TB_T - \int_0^T t dB_t \right) \\ &= \int_0^T 3B_t^2 dB_t + 3\int_0^T (T-t) dB_t \\ &= \int_0^T 3\left[ B_t^2 + (T-t) \right] dB_t. \end{aligned}$$

# *Proof :*

We know that

$$F = E[F] + \sum_{n=0}^{\infty} I_n(f_n).$$

Then, it suffices to write, for each $n \geq 1$,

$$I_n(f_n) = n! \int_0^T u_n(t) dB_t,$$

where

$$u_n(t) = \int_0^t \int_0^{t_{n-1}} \cdots \int_0^{t_1} f(t, t_1, t_n, \dots, t_{n-1}) dB_{t_1} \cdots dB_{t_{n-1}},$$

and take $u_t = \sum_{n=1}^{\infty} u_n(t)$. $\square$