# Introduction to Malliavin calculus

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Escuela de Probabilidad
CIMAT, Guanajuato, Septiembre 2016

# Introuduction

- Paul Malliavin (1925-2010) introduced in the 70's a calculus of variations with respect to the trajectories of Brownian motion, in order to provide a probabilistic proof of Hörmander's hypoellipticity theorem.

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- This calculus was further developed by Bismut, Stroock, Kusuoka and Watanabe, among others.
- The main application of this calculus is to show the existence and smoothness of densities of functionals of Gaussian processes.
- In combination with Stein's method, the Malliavin calculus has been recently used to establish quantitative results on normal approximations (Nourdin and Peccati).

# Finite dimensional case

The probability space $(\Omega, \mathcal{F}, P)$ is

- $\Omega = \mathbb{R}^n, \quad \mathcal{F} = \mathcal{B}(\mathbb{R}^n)$,
- $P$ is the Gaussian probability with density $p(x) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}$.

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*Differential operators :*

(i) Derivative operator :

$$\nabla F = \left( \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right),$$

where $F : \mathbb{R}^n \to \mathbb{R}$.

(ii) Divergence operator :

$$\delta(u) = \sum_{i=1}^{n} \left( u_i x_i - \frac{\partial u_i}{\partial x_i} \right) = \langle u, x \rangle - \mathrm{div} u,$$

where $u : \mathbb{R}^n \to \mathbb{R}^n$.

## Proposition

*$\delta$ is the adjoint of $\nabla$, that is,*

$$E(\langle u, \nabla F\rangle) = E(F\delta(u)),$$

*if $F$ and $u$ are continuously differentiable and their partial derivatives have at most polynomial growth.*

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*Proof :*
Integrating by parts, and using $\frac{\partial p}{\partial x_i} = -x_i p(x)$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \langle \nabla F, u\rangle p(x)dx &= \sum_{i=1}^{n} \int_{\mathbb{R}^n} \frac{\partial F}{\partial x_i} u_i p(x)dx \\ &= \sum_{i=1}^{n} \left( -\int_{\mathbb{R}^n} F \frac{\partial u_i}{\partial x_i} p(x)dx + \int_{\mathbb{R}^n} F u_i x_i p(x)dx \right) \\ &= \int_{\mathbb{R}^n} F\delta(u)p(x)dx.\end{aligned}$$

# The Wiener space

Consider the probability space $(\Omega, \mathcal{F}, P)$, where

- $\Omega = C([0, T])$
- $\mathcal{F}$ is the Borel $\sigma$-field of $\Omega$
- $P$ is the law of the Brownian motion (*Wiener measure*) : For all $0 \leq t_1 < \cdots < t_n$, $a_i < b_i$,

$$P\{\omega : a_i \leq \omega(t_i) \leq b_i, 1 \leq i \leq n\} = \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} e^{-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}} dx_1 \cdots dx_n,$$

with the convention $t_0 = 0$ and $x_0 = 0$.

# Derivative operator

Let $F : \Omega \to \mathbb{R}$. The derivative $DF$ takes values in $H = L^2([0, T])$. That is, $\{D_t F, T \in [0, T]\}$ is a stochastic process.

- For any $h \in H$, we denote by $B(h)$ the Wiener integral $B(h) = \int_0^T h(t)dB(t)$.
- Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f(B(h_1)\ldots, B(h_n)),$$

where $f \in C_p^\infty(\mathbb{R}^n)$ ($f$ and all ist partial derivatives have polynomial growth) and $h_i \in H$.

## Definition

For $F \in \mathcal{S}$ the derivative $DF$ is the $H$-valued random variable defined by

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(h_1)\ldots, B(h_n))h_i(t).$$

*Examples* : $D(B(h)) = h$, $D(B_{t_1}) = \mathbf{1}_{[0,t_1]}$.

- The Cameron-Martin space $H^1 \subset \Omega$ is the set of functions of the form $\psi(t) = \int_0^t h(s)ds$, where $h \in H$.
- For ant $h \in H$, $\langle DF, h \rangle_H$ is the derivative of $F$ in the direction of $\int_0^{\cdot} h(s)ds$ :

$$\langle DF, h \rangle_H = \int_0^T h_t D_t F dt = \frac{d}{d\epsilon} F \left( \omega + \epsilon \int_0^{\cdot} h_s ds \right) |_{\epsilon=0}.$$

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*Example :* If $F = B_{t_1}$

$$F\left(\omega + \epsilon \int_0^{\cdot} h_s ds\right) = \omega(t_1) + \epsilon \int_0^{t_1} h_s ds,$$

so, $\langle DF, h\rangle_H = \int_0^{t_1} h_s ds$, and $D_t F = \mathbf{1}_{[0,t_1]}(t)$.

# Divergence operator

Let $u \in \mathcal{S}_H$ be an smooth and cylindrical stochastic process of the form

$$u(t) = \sum_{j=1}^{n} F_j h_j(t),$$

where the $F_j \in \mathcal{S}$, and $h_j \in H$.

## Definition

We define the divergence of $u$, $\delta(u)$ as the random variable

$$\delta(u) = \sum_{j=1}^{n} F_j B(h_j) - \sum_{j=1}^{n} \langle DF_j, h_j \rangle_H .$$

*Example* : $\delta(h) = B(h)$.

## Proposition

*Let $F \in \mathcal{S}$ and $u \in \mathcal{S}_H$. Then,*

$$E(F\delta(u)) = E(\langle DF, u\rangle_H).$$

*Proof :* We can assume that

$$F = f(B(h_1) \ldots, B(h_n))$$

and

$$u = \sum_{j=1}^{n} g(B(h_1) \ldots, B(h_n))h_j,$$

where $h_1, \ldots, h_n$ are orthonormal elements in $H$. In this case, the duality relationship reduces to the finite dimensional case.

Notation : $D_h F = \langle DF, h \rangle_H$, for any $h \in H$ and $F \in \mathcal{S}$.

## Proposition

*Suppose that $u, v \in \mathcal{S}_H$, $F \in \mathcal{S}$ and $h \in H$. Then, if $\{e_i, i \geq 1\}$ is a complete orthonormal system in $H$ we have*

$$E(\delta(u)\delta(v)) = E(\langle u, v \rangle_H) + E\left(\sum_{i,j=1}^{\infty} D_{e_i}\langle u, e_j \rangle_H D_{e_j}\langle v, e_i \rangle_H\right), \quad (1)$$

$$D_h(\delta(u)) = \delta(D_h u) + \langle h, u \rangle_H, \quad (2)$$

$$\delta(Fu) = F\delta(u) - \langle DF, u \rangle_H. \quad (3)$$

- Property (1) can also be written as

$$E[\delta(u)\delta(v)] = E\left[\int_0^T u_t v_t dt\right] + E\left[\int_0^T \int_0^T D_s u_t D_t v_s ds dt\right].$$

## *Proof of* $\boxed{D_h(\delta(u)) = \delta(D_h u) + \langle h, u \rangle_H}$ *:*

Assume $u = \sum_{j=1}^n F_j h_j$. Then, using $D_h(B(h_j) = \langle h, h_j \rangle_H$, we obtain

$$\begin{aligned} D_h(\delta(u)) &= D_h \left( \sum_{j=1}^n F_j B(h_j) - \sum_{j=1}^n \langle DF_j, h_j \rangle_H \right) \\ &= \sum_{j=1}^n F_j \left\langle h, h_j \right\rangle_H + \sum_{j=1}^n \left( D_h F_j B(h_j) - \left\langle D_h \left( DF_j \right), h_j \right\rangle_H \right) \\ &= \langle u, h \rangle_H + \delta(D_h u). \end{aligned}$$

*Proof of* $\boxed{E(\delta(u)\delta(v)) = E(\langle u,v\rangle_H) + E\left(\sum_{i,j=1}^{\infty} D_{e_i}\langle u,e_j\rangle_H D_{e_j}\langle v,e_i\rangle_H\right)}$ *:*

Using the duality formula and property (2) yields

$$\begin{aligned}
E\left(\delta(u)\delta(v)\right)\right) &= E\left(\langle v, D(\delta(u))\rangle_H\right) \\
&= E\left(\sum_{i=1}^{\infty}\langle v,e_i\rangle_H D_{e_i}(\delta(u))\right) \\
&= E\left(\sum_{i=1}^{\infty}\langle v,e_i\rangle_H \left(\langle u,e_i\rangle_H + \delta(D_{e_i}u)\right)\right) \\
&= E\left(\langle u,v\rangle_H\right) + E\left(\sum_{i,j=1}^{\infty} D_{e_i}\left\langle u,e_j\right\rangle_H D_{e_j}\left\langle v,e_i\right\rangle_H\right).
\end{aligned}$$

# Proof of $\boxed{\delta(Fu) = F\delta(u) - \langle DF, u\rangle_H}$ :

For any smooth random variable $G \in \mathcal{S}$ we have,

$$D(FG) = FDG + GDF.$$

Then, using the duality relationship

$$\begin{aligned} E[\delta(Fu)G] &= E\left(\langle DG, Fu\rangle_H\right) \\ &= E\left(\langle u, D(FG) - GDF\rangle_H\right) \\ &= E\left(\left(\delta(u)F - \langle u, DF\rangle_H\right) G\right). \end{aligned}$$

This implies the result because $\mathcal{S}$ is dense in $L^2(\Omega)$.

## Proposition

*The operator $D$ is closable from $L^p(\Omega)$ to $L^p(\Omega; H)$ for any $p \geq 1$.*

*Proof :* Assume $F_N \in \mathcal{S}$ satisfies

$$F_N \overset{L^p(\Omega)}{\longrightarrow} 0, \quad DF_N \overset{L^p(\Omega;H)}{\longrightarrow} \eta.$$

Then, $\eta = 0$. Indeed, for any $u = \sum_{j=1}^N G_j h_j \in \mathcal{S}_H$ such that $G_j$, $G_j W(h_j)$ and $DG_j$ are bounded, we have

$$\begin{aligned} E(\langle \eta, u \rangle_H) &= \lim_{N\to\infty} E(\langle DF_N, u \rangle_H) \\ &= \lim_{N\to\infty} E(F_N \delta(u)) = 0. \end{aligned}$$

This implies that $\eta = 0$.

- For $p \geq 1$, we denote by $\mathbb{D}^{1,p}$ the closure of $\mathcal{S}$ with respect to the seminorm

$$\|F\|_{1,p} = \left( E[|F|^p] + E\left[ \left| \int_0^T (D_t F)^2 dt \right|^{p/2} \right] \right)^{1/p}.$$

- For $p = 2$, the space $\mathbb{D}^{1,2}$ is a Hilbert space with the scalar product

$$\langle F, G \rangle_{1,2} = E(FG) + E\left[ \int_0^T D_t F D_t G dt \right].$$

- In the same way we can introduce the spaces $\mathbb{D}^{1,p}(H)$ by taking the closure of $\mathcal{S}_H$.

## Definition

The domain of the divergence operator $\mathrm{Dom}\delta$ in $L^2(\Omega)$ is the set of processes $u \in L^2(\Omega \times [0, T])$ such that there exists $\delta(u) \in L^2(\Omega)$ satisfying the duality relationship

$$E(\langle DF, u\rangle_H) = E(\delta(u)F),$$

for any $F \in \mathbb{D}^{1,2}$.

- Clearly if $u_n \in \mathcal{S}_H$ satisfies $u_n \stackrel{L^2(\Omega;H)}{\longrightarrow} u$ and $\delta(u_n) \stackrel{L^2(\Omega)}{\longrightarrow} G$, then $u$ belongs to $\mathrm{Dom}\delta$ and $\delta(u) = G$.

- Property (1) holds for $u, v \in \mathbb{D}^{1,2}(H) \subset \mathrm{Dom}\delta$ and

$$E(\delta(u)^2) \leq E\left[\int_0^T (u_t)^2 dt\right] + E\left[\int_0^T \int_0^T (D_s u_t)^2 ds dt\right] = \|u\|_{1,2,H}^2.$$

- Property (2) :

$$\boxed{D_h(\delta(u)) = \delta(D_h u) + \langle h, u \rangle_H}$$

  holds if $u \in \mathbb{D}^{1,2}(H)$ and $D_h u$ is in the domain of $\delta$.

- Property (3) :

$$\boxed{\delta(Fu) = F\delta(u) - \langle DF, u \rangle_H}$$

  holds if $F \in \mathbb{D}^{1,2}$, $Fu \in L^2(\Omega; H)$, $u \in \mathrm{Dom}\delta$ and the right-hand side is square integrable.

# Meyer inequalities

**Theorem**

*For any* $p > 1$ *and* $u \in \mathbb{D}^{1,p}(H)$,

$$\boxed{E(|\delta(u)|^p) \le c_p \left( E(\|Du\|^p_{H\otimes H}) + E(\|u\|^p_H) \right)}.$$

1. Proved first by Paul-André Meyer (1934-2003).
2. A more modern proof is based on the boundedness in $L^p$ of the Riesz transform (Gilles Pisier).

# Iterated derivative

- The $k$th derivative $D^k F$ of a random variable $F \in \mathcal{S}$ is a $k$-parameter process obtained by iteration :

$$D^k_{t_1,\dots t_k} F = \sum_{i_1,\dots,i_k=1}^{n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} f(B(h_1), \dots B(h_n)) h_{i_1}(t_1) \cdots h_{i_k}(t_k).$$

- For any $p \geq 1$, the operator $D^k$ is closable from $L^p(\Omega)$ into $L^p(\Omega; H^{\otimes k})$ and we denote by $\mathbb{D}^{k,p}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p} = \left( E[|F|^p] + E\left[ \sum_{j=1}^{k} \left| \int_{[0,T]^j} (D^j_{t_1,\dots,t_j} F)^2 dt_1 \cdots dt_j \right|^{p/2} \right] \right)^{1/p}.$$

## Theorem (Gaveau-Trauber 1982)

*$L^2_T(\mathcal{P}) \subset \mathrm{Dom}\delta$ and for any $u \in L^2_T(\mathcal{P})$, $\delta(u)$ coincides with the Itô's stochastic integral :*

$$\boxed{\delta(u) = \int_0^T u_t dW_t}$$

- If $u$ not adapted $\delta(u)$ coincides with an anticipating stochastic integral introduced by Skorohod in 1975.
- Using techniques of Malliavin calculus, Nualart-Pardoux 1988 developed a stochastic calculus for the Skorohod integral.

*Proof :*

(i) If $u = \sum_{j=1}^n F_j \mathbf{1}_{[a_j, b_j]}$ and $F_j$ is a random variable in $\mathcal{S}$, $\mathcal{F}_{a_j}$-measurable, then $\delta(u)$ coincides with the Itô integral of $u$ because

$$\delta(u) = \sum_{j=1}^n F_j(B(b_j) - B(a_j)) - \sum_{j=1}^n \int_{a_j}^{b_j} D_t F dt = \sum_{j=1}^n F_j(B(b_j) - B(a_j)),$$

and $D_t F = 0$ if $t > a_j$.

(ii) The result follows by approximating any square integrable adapted process by cylindrical adapted smooth processes.

- If $u$ and $v$ are adapted, then for $s < t$, $D_t v_s = 0$ and for $s > t$, $D_s u_t = 0$ and Property (1) leads to the isometry property

$$E[\delta(u)\delta(v)] = E\left[\int_0^T u_t v_t dt\right].$$

- If $u$ is an adapted process in $\mathbb{D}^{1,2}(H)$, then from Property (2) we obtain

$$D_t\left(\int_0^T u_s dB_s\right) = u_t + \int_t^T D_t u_s dB_s,$$

because $D_t u_s = 0$ if $t > s$.

# Clark-Ocone formula

## Theorem (Clark-Ocone formula)

*Let* $F \in \mathbb{D}^{1,2}$. *Then,*

$$F = E(F) + \int_0^T E(D_t F | \mathcal{F}_t) dB_t.$$

# Clark-Ocone formula

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*Let $F \in \mathbb{D}^{1,2}$. Then,*

$$F = E(F) + \int_0^T E(D_t F | \mathcal{F}_t) dB_t.$$

*Example* :
$F = B_t^3$. Then $D_s F = 3B_t^2 \mathbf{1}_{[0,t]}(s)$ and

$$E(D_s F | \mathcal{F}_s) = 3E[(B_t - B_s + B_s)^2 | \mathcal{F}_s] = 3[t - s + B_s^2].$$

Therefore,

$$B_t^3 = 3 \int_0^t [t - s + B_s^2] dB_s.$$

*Proof :*

- For any $v \in L_a^2$ we can write, using the duality relationship

$$E\left(F\int_0^T v_t dB_t\right) = E(F\delta(v)) = E\left(\int_0^T D_t F v_t dt\right)$$
$$= \int_0^T E[E(D_t F|\mathcal{F}_t)v_t]dt.$$

- If we assume that $F = E(F) + \int_0^T u_t dB_t$, then by the Itô isometry

$$E\left(F\int_0^T v_t dB_t\right) = \int_0^T E(u_t v_t)dt.$$

Comparing these two expressions we deduce that

$$u_t = E(D_t F|\mathcal{F}_t)$$

almost everywhere in $\Omega \times [0, T]$.

# Derivative operator on the Wiener chaos

Let $f \in L^2_s([0,T]^n)$. Then $I_n(f) \in \mathbb{D}^{1,2}$, and

$$\boxed{D_t I_n(f) = n I_{n-1}(f(\cdot, t))}$$

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*Proof :* Assume $f = g^{\otimes n}$, with $\|g\| = 1$. Let $\theta = \int_0^T g_t dB_t$. Then

$$\begin{aligned} D_t I_n(f) &= D_t(h_n(\theta)) = h_n'(\theta) D_t\theta = n h_{n-1}(\theta) g_t \\ &= n g_t I_{n-1}(g^{\otimes(n-1)}) = n I_{n-1}(f(\cdot,t)). \end{aligned}$$

# Derivative operator on the Wiener chaos

Let $f \in L_s^2([0,T]^n)$. Then $I_n(f) \in \mathbb{D}^{1,2}$, and

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- Moreover

$$\begin{aligned} E\int_0^T [D_t I_n(f)]^2 dt &= n^2 \int_0^T E[I_{n-1}(f(\cdot,t))^2] dt \\ &= n^2(n-1)! \int_0^T \|f(\cdot,t)\|^2_{L^2([0,T]^{n-1})} dt \\ &= nn! \|f\|^2_{L^2([0,T]^n)} \\ &= nE[I_n(f)^2]. \end{aligned}$$

## Proposition

*Let $F \in L^2(\Omega)$ with the Wiener chaos expansion $F = \sum_{n=0}^{\infty} I_n(f_n)$. Then $F \in \mathbb{D}^{1,2}$ if and only if*

$$E(\|DF\|_H^2) = \sum_{n=1}^{\infty} n n! \|f_n\|_{L^2([0,T]^n)}^2 < \infty,$$

*and in this case*

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)).$$

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$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)).$$

- If $F \in \mathbb{D}^{k,2}$, then

$$D^k_{t_1,\dots,t_k} F = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) I_{n-k}(f_n(\cdot, t_1, \dots, t_k)).$$

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- If $F \in \mathbb{D}^{k,2}$, then

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- As a consequence, if $F \in \mathbb{D}^{\infty,2} := \cap_k \mathbb{D}^{k,2}$, then (*Stroock's formula*)

$$\boxed{f_n = \frac{1}{n!} E(D^n F)}$$

*Example :* $F = B_1^3$. Then,

$$f_1(t_1) = E(D_{t_1} B_1^3) = 3E(B_1^2)\mathbf{1}_{[0,1]}(t_1) = 3\mathbf{1}_{[0,1]}(t_1),$$
$$f_2(t_1, t_2) = \frac{1}{2}E(D^2_{t_1,t_2} B_1^3) = 3E(B_1)\mathbf{1}_{[0,1]}(t_1 \vee t_2) = 0,$$
$$f_3(t_1, t_2, t_3) = \frac{1}{6}E(D^3_{t_1,t_2,t_3} B_1^3) = \mathbf{1}_{[0,1]}(t_1 \vee t_2 \vee t_3),$$

and we obtain the Wiener chaos expansion

$$B_1^3 = 3B_1 + 6\int_0^1 \int_0^{t_1} \int_0^{t_2} dB_{t_1} dB_{t_2} dB_{t_3}.$$

# Divergence on the Wiener chaos expansion

A square integrable stochastic process $u = \{u_t, t \in [0, T]\}$, has an orthogonal expansion of the form

$$u_t = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t)),$$

where $f_0(t) = E[u_t]$ and for each $n \geq 1$, $f_n \in L^2([0, T]^{n+1})$ is a symmetric function in the first $n$ variables.

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**Proposition**

*The process u belongs to the domain of $\delta$ if and only if the series*

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\widetilde{f}_n) \tag{4}$$

*converges in $L^2(\Omega)$.*

*Proof :* Suppose that $G = I_n(g)$ is a multiple stochastic integral of order $n \geq 1$, where $g$ is symmetric. Then,

$$\begin{aligned} E(\langle u, DG \rangle_H) &= \int_0^T E(I_{n-1}(f_{n-1}(\cdot,t)) n I_{n-1}(g(\cdot,t))) dt \\ &= n(n-1)! \int_0^T \langle f_{n-1}(\cdot,t), g(\cdot,t) \rangle_{L^2([0,T]^{n-1})} dt \\ &= n! \langle f_{n-1}, g \rangle_{L^2([0,T]^n)} = n! \left\langle \widetilde{f}_{n-1}, g \right\rangle_{L^2([0,T]^n)} \\ &= E\left( I_n\left(\widetilde{f}_{n-1}\right) I_n(g) \right) = E\left( I_n\left(\widetilde{f}_{n-1}\right) G \right). \end{aligned}$$

If $u \in \mathrm{Dom}\delta$, we deduce that

$$E(\delta(u)G) = E\left( I_n\left(\widetilde{f}_{n-1}\right) G \right)$$

for every $G \in \mathcal{H}_n$. This implies that $I_n\left(\widetilde{f}_{n-1}\right)$ coincides with the projection of $\delta(u)$ on the $n$th Wiener chaos. Consequently, the series in (4) converges in $L^2(\Omega)$ and its sum is equal to $\delta(u)$. The converse can be proved by similar arguments.

# The Ornstein-Uhlenbeck semigroup

Consider the one-parameter semigroup $\{T_t, t \geq 0\}$ of contraction operators on $L^2(\Omega)$ defined by

$$T_t(F) = \sum_{n=0}^{\infty} e^{-nt} I_n(f_n),$$

where $F = \sum_{n=0}^{\infty} I_n(f_n)$.

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**Proposition (*Mehler's formula*)**

*Let $B' = \{B'_t, t \in [0, T]\}$ be an independent copy of $B$. Then, for any $t \geq 0$ and $F \in L^2(\Omega)$ we have*

$$T_t(F) = E'(F(e^{-t}B + \sqrt{1 - e^{-2t}}B')), \tag{5}$$

*where $E'$ denotes the mathematical expectation with respect to $B'$.*

*Proof :* Both $T_t$ and the right-hand side of (5) give rise to linear contraction operators on $L^2(\Omega)$. Thus, it suffices to show (5) when $F = \exp\left(\lambda B(h) - \frac{1}{2}\lambda^2\right)$, where $B(h) = \int_0^T h_t dB_t$ and $h \in H$ is an element of norm one, and $\lambda \in \mathbb{R}$. We have,

$$E' \left( \exp\left( e^{-t}\lambda B(h) + \sqrt{1-e^{-2t}}\lambda B'(h) - \frac{1}{2}\lambda^2 \right)\right)$$
$$= \exp\left( e^{-t}\lambda B(h) - \frac{1}{2} e^{-2t}\lambda^2 \right) = \sum_{n=0}^{\infty} e^{-nt} \frac{\lambda^n}{n!} h_n\left(B(h)\right) = T_t F,$$

because

$$F = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} h_n\left(B(h)\right)$$

and

$$e^{az - \frac{1}{2}a^2} = \sum_{n=0}^{\infty} \frac{a^n}{n!} h_n(z).$$

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2. The oerator $T_t$ is symmetric :

$$E(GT_t(F)) = E(FT_t(G)) = \sum_{n=0}^{\infty} e^{-nt} E(I_n(f_n) I_n(g_n)).$$

3. *Hypercontractivity* : If $F \in L^p(\Omega)$, $p > 1$ and $q(t) = e^{2t}(p-1) + 1 > p$, $t > 0$, then

$$\|T_t F\|_{q(t)} \leq \|F\|_p,$$

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   Consequences :

   - For any $1 < p < q < \infty$ the norms $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent on any Wiener chaos $\mathcal{H}_n$.
   - For each $n \geq 1$ and $1 < p < \infty$, the projection on the $n$th Wiener chaos is bounded in $L^p(\Omega)$.

# The generator of the Ornstein-Uhlenbeck semigroup

- The infinitesimal generator of the semigroup $T_t$ in $L^2(\Omega)$ is given by

$$LF = \lim_{t \downarrow 0} \frac{T_t F - F}{t} = \sum_{n=1}^{\infty} -n I_n(f_n),$$

if $F = \sum_{n=0}^{\infty} I_n(f_n)$.

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  if $F = \sum_{n=0}^{\infty} I_n(f_n)$.
- The domain of $L$ is

$$\mathrm{Dom}\, L = \{F \in L^2(\Omega), \ \sum_{n=1}^{\infty} n^2 n! \|f_n\|_2^2 < \infty\} = \mathbb{D}^{2,2}.$$

The next proposition explains the relationship between the operators $D$, $\delta$, and $L$.

## Proposition

*Let $F \in \mathbb{D}^{1,2}$. Then $F \in \mathrm{Dom}\, L$ if and only if $DF \in \mathrm{Dom}\, \delta$, and in this case, we have*

$$\boxed{\delta DF = -LF}$$

*Proof* Let $F = \sum_{n=0}^{\infty} I_n(f_n)$. For any random variable $G = I_m(g_m)$ we have, using the duality relationship

$$\begin{aligned} E(G\delta DF) &= E(\langle DG, DF\rangle_H) = mm!\langle g_m, f_m\rangle_{L^2([0,T]^n)} \\ &= E\left(G\sum_{n=1}^{\infty} nI_n(f_n)\right) = -E(GLF), \end{aligned}$$

and the result follows easily.

The operator $L$ behaves as a second-order differential operator on smooth random variables.

## Proposition

*Suppose that $F = (F^1, \dots, F^m)$ is a random vector whose components belong to $\mathbb{D}^{2,4}$. Let $\varphi$ be a function in $C^2(\mathbb{R}^m)$ with bounded first and second partial derivatives. Then $\varphi(F) \in \mathrm{Dom}\, L$, and*

$$L(\varphi(F)) = \sum_{i,j=1}^{m} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(F) \langle DF^i, DF^j \rangle_H + \sum_{i=1}^{m} \frac{\partial \varphi}{\partial x_i}(F) LF^i.$$

*Proof :* Suppose first that $F \in \mathcal{S}$ is of the form

$$F = f(B(h_1), \ldots, B(h_n)),$$

$f \in C_p^\infty(\mathbb{R}^n)$. Then

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(h_1), \ldots, B(h_n)) h_i(t).$$

Consequently, $DF \in \mathcal{S}_H \subset \mathrm{Dom}\,\delta$ and we obtain

$$\begin{aligned} \delta DF &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(h_1), \ldots, B(h_n)) B(h_i) \\ &\quad - \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(B(h_1), \ldots, B(h_n)) \langle h_i, h_j \rangle_H, \end{aligned}$$

which yields the desired result because $L = -\delta D$. In the general case, it suffices to approximate $F$ by smooth random variables in the norm $\| \cdot \|_{2,4}$, and $\varphi$ by functions in $C_p^\infty(\mathbb{R}^m)$, and use the continuity of the operator $L$ in the norm $\| \cdot \|_{2,2}$ .

- In the finite-dimensional case ($\Omega = \mathbb{R}^n$ equipped with the standard Gaussian law),

$$L = \Delta - x \cdot \nabla$$

coincides with the generator of the Ornstein-Uhlenbeck process $\{X_t, t \geq 0\}$ in $\mathbb{R}^n$, which is the solution to the stochastic differential equation

$$dX_t = \sqrt{2} dB_t - X_t dt,$$

where $\{B_t, t \geq 0\}$ is a standard $n$-dimensional Brownian motion.

# Gaussian processes

The Malliavin calculus can be developed for any Gaussian process :

- For a Brownian motion on $[0, T]$, $H = L^2([0, T])$ and $B(h) = \int_0^T h(t) dB_t$.
- In the previous example, $H = L^2([0, \infty); \mathbb{R}^d)$ and $B(h) = \sum_{i=1}^d \int_0^\infty h_k(t) dB_t^k$.
- More generally, we can consider a centered Gaussian family of random variables $B = \{B(h), h \in H\}$ with covariance

$$E(B(h)B(g)) = \langle h, g \rangle_H,$$

where $H$ is a Hilbert space. The process $B$ is called an *isonormal Gaussian process*.