Stein's method and normal approximations

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Stein's method for normal approximation

Lemma

A random variable Z has the $N(0,1)$ law if and only if for any $f \in C^1(\mathbb{R})$ with compact support,

$$E[f'(Z) - Zf(Z)] = 0 \tag{1}$$

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A random variable Z has the $N(0,1)$ law if and only if for any $f \in C^1(\mathbb{R})$ with compact support,

$$E[f'(Z) - Zf(Z)] = 0 \qquad (1)$$

Proof : Supppose Z has a C^1 density $p(x)$. Integrating by parts,

$$\begin{aligned} E[f'(Z) - Zf(Z)] &= \int_{\mathbb{R}} [f'(x)p(x) - xf(x)p(x)]dx \\ &= \int_{\mathbb{R}} [-p'(x) - xp(x)]f(x)dx. \end{aligned}$$

So (1) is equivalent to $p'(x) = -xp(x)$, that is, $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

- Let $Z \sim N(0,1)$, and fix a Borel function $h : \mathbb{R} \to \mathbb{R}$ such that $E[|h(Z)|] < \infty$. Stein's equation associated with h is

$$\boxed{f_h'(x) - xf_h(x) = h(x) - E(h(Z)),}$$

where we require f_h to be absolutely continuous with a version of f_h' satisfying this equation for each $x \in \mathbb{R}$.

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Proposition

The unique solution to Stein's equation satisfying $\lim_{x\to\pm\infty} e^{-x^2/2} f_h(x) = 0$ *is*

$$\begin{aligned} f_h(x) &= e^{x^2/2} \int_{-\infty}^{x} (h(y) - E[h(Z)]) e^{-y^2/2} dy \\ &= -e^{x^2/2} \int_{x}^{\infty} (h(y) - E[h(Z)]) e^{-y^2/2} dy. \end{aligned}$$

Definition

A class $\mathscr{H}$ of Borel functions is called *separating* if for any random variables F and G such that $E[h(F)] = E[h(G)]$ for all $h \in \mathscr{H}$, then F and G have the same law.

Definition

A class $\mathscr{H}$ of Borel functions is called *separating* if for any random variables F and G such that $E[h(F)] = E[h(G)]$ for all $h \in \mathscr{H}$, then F and G have the same law.

Definition

Let $\mathscr{H}$ be a separating class and let F and G be two random variables such that $h(F)$ and $h(G)$ are in $L^1(\Omega)$ for any $h \in \mathscr{H}$. Then we define

$$d_{\mathscr{H}}(F, G) = \sup\{|E[h(F)] - E[h(G)]| : h \in \mathscr{H}\}.$$

Proposition

Let F and Z be random variables such that Z is $N(0,1)$. If $\mathscr{H}$ is a separating class of functions such that $E[|h(Z)|] < \infty$ and $E[|h(F)|] < \infty$ for every $h \in \mathscr{H}$, then

$$\boxed{d_{\mathscr{H}}(F,Z) \le \sup_{h \in \mathscr{H}} |E[f_h'(F) - Ff_h(F)]|.}$$

Proof : From Stein's equation, we obtain putting $x = F$,

$$h(F) - E[h(Z)] = f_h'(F) - Ff(F),$$

and taking the expectation this yields

$$E[h(F)] - E[h(Z)] = E[f_h'(F) - Ff_h(F)],$$

which implies the result.

Examples :

1. The *Kolmogorov distance*

$$d_{Kol}(F,G) = \sup_{z\in\mathbb{R}} |P(F \le z) - P(G \le z)|$$

is obtained by choosing

$$\mathscr{H} = \{\mathbf{1}_{(-\infty,z]}, z \in \mathbb{R}\}.$$

In this case, if we set $f_z = f_{\mathbf{1}_{(-\infty,z]}}$, then

$$f_z(x) = \begin{cases} \sqrt{2\pi} e^{x^2/2} \Phi(x)[1-\Phi(z)], & \textit{if} \quad x \le z, \\ \sqrt{2\pi} e^{x^2/2} \Phi(z)[1-\Phi(x)], & \textit{if} \quad x \ge z, \end{cases}$$

which implies $\|f_z\|_\infty \le \frac{\sqrt{2\pi}}{4}$ and $\|f_z'\|_\infty \le 1$.

2. The *total variation distance*

$$d_{TV}(F,G) = \sup_{B \in \mathcal{B}(\mathbb{R})} |P(F \in B) - P(G \in B)|$$

corresponds to

$$\mathscr{H} = \{\mathbf{1}_B, B \in \mathcal{B}(\mathbb{R})\}.$$

In this case, $\|f_h\|_\infty \leq \sqrt{\frac{\pi}{2}}$ and $\|f_h'\|_\infty \leq 2$, for any $h \in \mathscr{H}$.

3. The *Wasserstein distance*, d_W corresponds to the class $\mathscr{H}$ of all Lipschitz functions with Lipschitz constant bounded by 1. In this case, f_h has the representation

$$f_h(x) = -\int_0^\infty \frac{e^{-t}}{\sqrt{1-e^{-2t}}} E[h(e^{-tx} + \sqrt{1-e^{-2t}}Z)Z]dt,$$

where $Z \sim N(0,1)$, which implies $f_h \in C^1$ and $\|f_h'\|_\infty < \sqrt{\frac{2}{\pi}}$.

- Set $d_1 = d_{Kol}$, $d_2 = d_{TV}$ and $d_3 = d_W$. Then, if F and Z are random variables such that Z is $N(0,1)$, we obtain

$$d_i(F,Z) \le \sup_{f \in \mathscr{F}_i} |E[f'(F)] - E[Ff(F)]|,$$

where

$$\mathscr{F}_i = \begin{cases} \{f \in C^1 : \|f\|_\infty \le \frac{\sqrt{2\pi}}{4}, \|f'\|_\infty \le 1\}, & \textit{if} \quad i = 1 \\ \{f \in C^1 : \|f\|_\infty \le \sqrt{\pi/2}, \|f'\|_\infty \le 2\}, & \textit{if} \quad i = 2 \\ \{f \in C^1 : \|f'\|_\infty \le \sqrt{2/\pi}\}, & \textit{if} \quad i = 3. \end{cases}$$

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- For d_1 and d_2 we can take $f \in C^1$, because for any function h, such that $\|h\|_\infty \leq 1$, we can find a sequence of continuous functions h_n bounded by 1, such that h_n converges to h almost everywhere with respect to the measure $\ell + (P \circ F^{-1})$.

Stein meets Malliavin

Let $\{B(h), h \in H\}$ be an isonormal Gaussian process.

Theorem (Nourdin-Peccati)

Suppose that $F \in \mathbb{D}^{1,2}$ satisfies $F = \delta(u)$, where u belongs to the domain in L^2 of the divergence operator δ. Let Z be a $N(0,1)$ random variable. Then,

$$\boxed{d_{TV}(F, Z) \le 2E[|1 - \langle DF, u\rangle_H|].}$$

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Proof : It follows from

$$E[Ff(F)] = E[\delta(u)f(F)] = E[\langle u, DF\rangle_H]$$

and, therefore,

$$|E[f'(F)] - E(Ff(F)]| = |E[f(F)[1 - \langle DF, u\rangle_H]|,$$

for any $f \in \mathscr{F}_i$.

Example

Let $B = \{B_t, t \int [0, T]\}$ be a Brownian motion. Suppose that $F = \int_0^T u_s dB_s$, where u is a progressively measurable process in $\mathbb{D}^{1,2}(H)$. Then,

$$D_t F = u_t + \int_t^T D_t u_s dB_s,$$

and

$$\langle u, DF \rangle_H = \|u\|_H^2 + \int_0^T \left(\int_t^T D_t u_s dB_s \right) u_t dt.$$

As a consequence,

$$\begin{aligned} d_{TV}(F, Z) &\leq 2E\left(|1 - \|u\|_H^2|\right) + 2E\left(\left| \int_0^T \left(\int_t^T D_t u_s dB_s \right) u_t dt \right| \right) \\ &\leq 2E\left(|1 - \|u\|_H^2|\right) + 2\left[E \int_0^T \left(\int_0^s u_t D_t u_s dt \right)^2 ds \right]^{\frac{1}{2}}. \end{aligned}$$

Therefore, a sequence $F_n = \int_0^T u_s^{(n)} dB_s$, where $u^{(n)} \in \mathbb{D}^{1,2}(H)$, converges in law to $N(0,1)$ (even in total variation) if :

(i) $\|u^{(n)}\|_H^2 \to 1$ in $L^1(\Omega)$ and

(ii) $E \int_0^T \left(\int_0^s u_t^{(n)} D_t u_s^{(n)} dt \right)^2 ds \to 0.$

Example : $u_t^{(n)} = \sqrt{2n} t^n \exp(B_t(1-t)) \mathbf{1}_{[0,1]}(t).$

- We can take $u = -DL^{-1}F$, because

$$F = LL^{-1}F = -\delta DLL^{-1}F,$$

and, we obtain

$$\boxed{d_{TV}(F,Z) \le 2E[|1 - \langle DF, -DL^{-1}F\rangle_H|]}$$

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- If $E[F^2] = \sigma^2 > 0$ and we take $Z \sim N(0, \sigma^2)$, we can derive the following inequality :

$$d_{TV}(F, Z) \leq \frac{2}{\sigma^2} E[|\sigma^2 - \langle DF, u\rangle_H|].$$

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- There are also multidimensional versions of these results.

Normal approximation on a fixed Wiener chaos

Recall that for any $F \in \mathbb{D}^{1,2}$ such that $E[F] = 0$,

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Proposition

Suppose $F \in \mathcal{H}_q$ for some $q \geq 2$ and $E(F^2) = \sigma^2$. Then,

$$\boxed{d_{TV}(F, Z) \leq \frac{2}{q\sigma^2} \sqrt{\mathrm{Var}\left(\|DF\|_H^2\right)}}.$$

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$$\boxed{d_{TV}(F, Z) \leq \frac{2}{q\sigma^2} \sqrt{\mathrm{Var}\left(\|DF\|_H^2\right)}}.$$

Proof : Using $L^{-1}F = -\frac{1}{q}F$ and $E[\|DF\|_H^2] = q\sigma^2$, we obtain

$$\begin{aligned} E[|\sigma^2 - \langle DF, -DL^{-1}F \rangle_H|] &= E\left[\left|\sigma^2 - \frac{1}{q}\|DF\|_H^2\right|\right] \\ &\leq \frac{1}{q}\sqrt{\mathrm{Var}\left(\|DF\|_H^2\right)}. \end{aligned}$$

Proposition

Suppose that $F = I_q(f) \in \mathcal{H}_q$, $q \geq 2$. Then,

$$\mathrm{Var}\left(\|DF\|_H^2\right) \leq \frac{(q-1)q}{3}(E(F^4) - 3\sigma^4) \leq (q-1)\mathrm{Var}\left(\|DF\|_H^2\right).$$

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Proof : This proposition is a consequence of the following two formulas :

FORMULA 1 :

$$\operatorname{Var}\left(\|DF\|_H^2\right) = \sum_{r=1}^{q-1} r^2 (r!)^2 \binom{q}{r}^4 (2q-2r)! \|f \widetilde{\otimes}_r f\|_{H^{\otimes(2q-2r)}}^2$$

FORMULA 1 :

$$\boxed{\mathrm{Var}\left(\|DF\|_H^2\right) = \sum_{r=1}^{q-1} r^2 (r!)^2 \binom{q}{r}^4 (2q-2r)! \|f \widetilde{\otimes}_r f\|^2_{H^{\otimes(2q-2r)}}}$$

In fact, we have $D_t F = q I_{q-1}(f(\cdot,t))$, and using the product formula for multiple stochastic integrals we obtain

$$\begin{aligned}
\|DF\|_H^2 &= q^2 \int_0^T I_{q-1}(f(\cdot,t))^2 dt \\
&= q^2 \sum_{r=0}^{q-1} r! \binom{q-1}{r}^2 I_{2q-2r-2}(f \widetilde{\otimes}_{r+1} f) \\
&= q^2 \sum_{r=1}^{q} (r-1)! \binom{q-1}{r-1}^2 I_{2q-2r}(f \widetilde{\otimes}_r f) \\
&= q q! \|f\|^2_{H^{\otimes q}} + q^2 \sum_{r=1}^{q-1} (r-1)! \binom{q-1}{r-1}^2 I_{2q-2r}(f \widetilde{\otimes}_r f). \qquad (2)
\end{aligned}$$

Formula 1 follows from the isometry property of multiple integrals.

FORMULA 2 :

$$E[F^4] - 3\sigma^4 = \frac{3}{q} \sum_{r=1}^{q-1} r(r!)^2 \binom{q}{r}^4 (2q-2r)! \|f \widetilde{\otimes}_r f\|^2_{H^{\otimes(2q-2r)}}$$

FORMULA 2 :

$$E[F^4] - 3\sigma^4 = \frac{3}{q} \sum_{r=1}^{q-1} r(r!)^2 \binom{q}{r}^4 (2q-2r)! \| f \widetilde{\otimes}_r f \|^2_{H^{\otimes(2q-2r)}}$$

Using that $-L^{-1}F = \frac{1}{q}F$ and $L = -\delta D$ we can write

$$\begin{aligned} E[F^4] &= E[F \times F^3] = E[(-\delta D L^{-1} F) F^3] = E[\langle -DL^{-1}F, D(F^3)\rangle_H] \\ &= \frac{1}{q} E[\langle DF, D(F^3)\rangle_H] = \frac{3}{q} E[F^2 \|DF\|^2_H]. \end{aligned} \tag{3}$$

By the product formula of multiple integrals,

$$F^2 = I_q(f)^2 = q! \|f\|^2_{H^{\otimes q}} + \sum_{r=0}^{q-1} r! \binom{q}{r}^2 I_{2q-2r}(f \tilde{\otimes}_r f). \tag{4}$$

Formula 2 follows from (3), (4), (2) and the isometry property of multiple integrals.

Fourth Moment theorem

Stein's method combined with Malliavin calculus leads to a simple proof of the Fourth Moment theorem :

Theorem (N.-Peccati '05, N.-Ortiz '07)

Fix $q \geq 2$. *Let* $F_n = I_q(f_n) \in \mathcal{H}_q$, $n \geq 1$ *be such that*

$$\lim_{n\to\infty} E(F_n^2) = \sigma^2.$$

The following conditions are equivalent :

(i) $F_n \Rightarrow N(0, \sigma^2)$, *as* $n \to \infty$.

(ii) $E(F_n^4) \to 3\sigma^4$, *as* $n \to \infty$.

(iii) $\|DF_n\|_H^2 \to q\sigma^2$ *in* $L^2(\Omega)$, *as* $n \to \infty$.

(iv) *For all* $1 \leq r \leq q-1$, $f_n \otimes_r f_n \to 0$, *as* $n \to \infty$.

- This theorem constitutes a drastic simplification of the method of moments.

Proof :

1. (i) implies (ii) because for any $p > 2$, the hypercontractivity property of the Ornstein-Uhlenbeck semigroup implies

$$\sup_n \|F_n\|_p \leq (p-1)^{\frac{n}{2}} \sup_n \|F_n\|_2 < \infty.$$

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2. The equivalence of (ii) and (iii)) follows from the previous proposition, and these conditions imply (i), with convergence in total variation.
3. (iv) implies (ii) and (iii) because $\|f_n \tilde{\otimes}_r f_n\| \leq \|f_n \otimes_r f_n\|$.

4. Let us show that (ii) implies (iv). From (4) we get

$$
\begin{aligned}
E[F_n^4] &= \sum_{r=0}^{q} (r!)^2 \binom{q}{r}^2 (2q-2r)! \|f_n \tilde{\otimes}_r f_n\|^2_{H^{\otimes(2q-2r)}} \\
&= (2q)! \|f_n \tilde{\otimes} f_n\|^2_{H^{\otimes 2q}} + \sum_{r=1}^{q-1} (r!)^2 \binom{q}{r}^2 (2q-2r)! \|f_n \tilde{\otimes}_r f_n\|^2_{H^{\otimes(2q-2r)}} \\
&\quad + (q!)^2 \|f_n\|^4_H.
\end{aligned}
$$

Then, we use the fact that $(2q)! \|f_n \tilde{\otimes} f_n\|^2_{H^{\otimes 2q}}$ equals to $2(q!)^2 \|f_n\|^4_H$ plus a linear combination of the terms $\|f_n \otimes_r f_n\|^2_{H^{\otimes(2q-2r)}}$, with $1 \leq r \leq q-1$, to conclude that

$$
\|f_n \otimes_r f_n\|_{H^{\otimes(2q-2r)}} \to 0, \quad 1 \leq r \leq q-1.
$$

Multivariate Gaussian approximation

- Let $B = \{B_t, t \in [0, T]\}$ be a Brownian motion.

Theorem (Peccati-Tudor '05)

Let $d \geq 2$ and $q_1, \dots, q_d \geq 1$. Consider random vectors

$$F_n = (F_{1,n}, \dots, F_{d,n}) = (I_{q_1}(f_{1,n}), \dots, I_{q_d}(f_{d,n})),$$

where $f_{i,n} \in L^2_s([0,T]^i)$. Suppose that, for any $1 \leq i, j \leq d$,

$$\lim_{n \to \infty} E(F_{i,n} F_{j,n}) = C_{i,j}.$$

Then, the following two conditions are equivalent :

(i) $F_n \Rightarrow N_d(0, C)$.

(ii) For every $i = 1, \dots, d$, $F_{i,n} \Rightarrow N(0, C_{i,i})$.

- In particular, if $q_1 < \cdots < q_d$, then C is diagonal and F_n converges in law to a Gaussian random vector with independent components.

Chaotic Central Limit Theorem

Theorem (Hu-N. '05)

Let $F_n = \sum_{q=1}^{\infty} I_q(f_{q,n})$, $n \geq 1$. *Suppose that :*

(i) *For all* $q \geq 1$, $q! \|f_{q,n}\|^2 \to \sigma_q^2$ *as* $n \to \infty$.

(ii) *For all* $q \geq 2$ *and* $1 \leq r \leq q-1$, $f_{q,n} \otimes_r f_{q,n} \to 0$ *as* $n \to \infty$.

(iii) $q! \|f_{q,n}\|^2 \leq \delta_q$, *where* $\sum_q \delta_q < \infty$.

Then, as n tends to infinity

$$F_n \Rightarrow N(0, \sigma^2), \quad \text{where} \quad \sigma^2 = \sum_{q=1}^{\infty} \sigma_q^2.$$

- Assuming (i), condition (ii) is equivalent to (ii)' : $\lim_{n\to\infty} E(I_q(f_{q,n})^4) = 3\sigma_q^4$, $q \geq 2$.
- The theorem implies the convergence in law of the whole sequence $(I_q(f_{q,n}), q \geq 1)$ to an infinite dimensional Gaussian vector with independent components.

Breuer-Major theorem

- A function $f \in L^2(\mathbb{R}, \gamma)$, where $\gamma = N(0,1)$, has *Hermite rank* $d \geq 1$ if

$$f(x) = \sum_{q=d}^{\infty} a_q h_q(x),$$

  and $a_d \neq 0$. *Example :* $f(x) = |x|^p - \int_{\mathbb{R}} |x|^p d\gamma(x)$ has Hermite rank 1 if $p > 0$ is not an even integer.

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- Let $X = \{X_k, k \in \mathbb{Z}\}$ be a centered stationary sequence with unit variance. Set $\rho(v) = E[X_0 X_v]$ for $v \in \mathbb{Z}$.

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Theorem (Breuer-Major '83)

Let $f \in L^2(\mathbb{R}, \gamma)$ with Hermite rank $d \geq 1$ and assume $\sum_{v \in \mathbb{Z}} |\rho(v)|^d < \infty$. Then,

$$V_n := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(X_k) \Rightarrow N(0, \sigma^2),$$

as $n \to \infty$, where $\sigma^2 = \sum_{q=d}^{\infty} q! a_q^2 \sum_{v \in \mathbb{Z}} \rho(v)^q$.

Sketch of the proof :

- From the chaotic Central Limit Theorem, it suffices to consider the case $f = a_q h_q$, $q \geq d$.

Sketch of the proof :

- From the chaotic Central Limit Theorem, it suffices to consider the case $f = a_q h_q$, $q \geq d$.
- There exists a sequence $\{e_k, k \geq 1\}$ in $H = L^2([0, T])$ such that

$$\langle e_k, e_j \rangle_H = \rho(k - j).$$

The sequence $\{B(e_k)\}$ has the same law as $\{X_k\}$, and we may replace V_n by

$$G_n = \frac{a_q}{\sqrt{n}} \sum_{k=1}^{n} h_q(B(e_k)) = I_q(f_{q,n}),$$

where $f_{q,n} = \frac{a_q}{\sqrt{n}} \sum_{k=1}^{n} e_k^{\otimes q}$.

- We can write

$$q!\|f_{q,n}\|^2_{H^{\otimes q}} = \frac{q!a_q^2}{n}\sum_{k,j=1}^{n}\rho(k-j)^q = q!a_q^2\sum_{v\in\mathbb{Z}}\rho(v)^q\left(1-\frac{|v|}{n}\right)\mathbf{1}_{\{|v|<n\}},$$

and by the dominated convergence theorem

$$E[G_n^2] = q!\|f_{q,n}\|^2_{H^{\otimes q}} \to q!a_q^2\sum_{v\in\mathbb{Z}}\rho(v)^q = \sigma^2.$$

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$$E[G_n^2] = q!\|f_{q,n}\|^2_{H^{\otimes q}} \to q!a_q^2\sum_{v\in\mathbb{Z}}\rho(v)^q = \sigma^2.$$

- Applying the Fourth Moment Theorem, It suffices to show that for $r=1,\dots,q-1$,

$$f_{q,n}\otimes_r f_{q,n} = \frac{a_q}{n}\sum_{k,j=1}^{n}\rho(k-j)^r e_k^{\otimes(q-r)}\otimes e_j^{\otimes(q-r)} \to 0.$$

We have

$$\|f_{q,n}\otimes_r f_{q,n}\|^2_{H^{\otimes(2q-2r)}} = \frac{a_q^4}{n^2}\sum_{i,j,k,\ell=1}^{n}\rho(k-j)^r\rho(i-\ell)^r\rho(k-i)^{q-r}\rho(j-\ell)^{q-r}.$$

- Using $|\rho(k-j)^r \rho(k-i)^{q-r}| \le |\rho(k-j)|^q + |\rho(k-i)|^q$, we obtain

$$\begin{aligned}\|f_{q,n} \otimes_r f_{q,n}\|^2_{H^{\otimes(2q-2r)}} &\le 2a_q^4 \sum_{k\in\mathbb{Z}} |\rho(k)|^q \left(n^{-1+\frac{r}{q}} \sum_{|i|\le n} |\rho(i)|^r \right) \\ &\quad \times \left(n^{-1+\frac{q-r}{q}} \sum_{|j|\le n} |\rho(j)|^{q-r} \right).\end{aligned}$$

Then, it suffices to show that for $r = 1, \dots, q-1$,

$$n^{-1+\frac{r}{q}} \sum_{|i|\le n} |\rho(i)|^r \to 0.$$

- This follows from Hölder's inequality. Indeed, for a fixed $\delta \in (0,1)$, we have the estimates

$$n^{-1+\frac{r}{q}} \sum_{|i| \le [n\delta]} |\rho(i)|^r \le n^{-1+\frac{r}{q}} (2[n\delta]+1)^{1-\frac{r}{q}} \left(\sum_{i \in \mathbb{Z}} |\rho(i)|^q \right)^{\frac{r}{q}} \le c\delta^{1-\frac{r}{q}},$$

and

$$n^{-1+\frac{r}{q}} \sum_{[n\delta] < |i| \le n} |\rho(i)|^r \le \left(\sum_{[n\delta] < |i| \le n} |\rho(i)|^q \right)^{\frac{r}{q}}.$$

The first term converges to zero as δ tends to zero and the second one converges to zero for fixed δ as $n \to \infty$.

Berry-Esséen asymptotics

- The following exact asymptotic behavior was proved in Nourdin-Peccati '10 :

$$[P(F_n \le z) - P(Z \le z)] \sim \varphi(n) \frac{\rho}{3q} \Phi^{(3)}(z),$$

as $n \to \infty$, where :

(i) $F_n \in \mathcal{H}_q$, $E(F_n^2) \to 1$,

(ii) $\varphi(n) = \sqrt{E\left[\left(1 - \frac{1}{q}\|DF_n\|_H^2\right)^2\right]}$,

(iii) $\rho = \lim_n E(F_n \|DF_n\|_H^2)$, and $\Phi(z) = P(Z \le z)$.