Fractional Brownian motion

The fractional Brownian motion (fBm) $B^H = \{B_t^H, t \geq 0\}$ is a zero mean Gaussian process with covariance

$$E(B_s^H B_t^H) = R_H(s,t) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

$H \in (0,1)$ is called the Hurst parameter.

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- $E(B_t^H - B_s^H)^2 = |t-s|^{2H}$.
- For any $\gamma < H$, with probability one, the trajectories $t \to B_t^H(\omega)$ are Hölder continuous of order γ :

$$|B_t^H(\omega) - B_s^H(\omega)| \leq G_{\gamma,T}(\omega)|t-s|^{\gamma}, \quad s,t \in [0,T].$$

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- For $H = \frac{1}{2}$, $B^{\frac{1}{2}}$ is a Brownian motion.

Self-similarity :

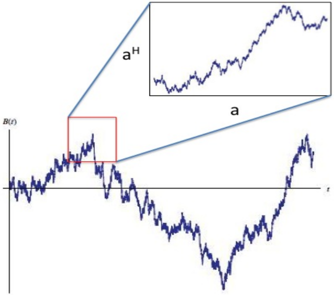
For all $a > 0$, the processes

$$\{a^{-H} B^H_{at}, t \geq 0\}$$

and

$$\{B^H_t, t \geq 0\}$$

have the same probability distribution (they are fractional Brownian motions with Hurst parameter H).



- *Correlated increments :*
  (i) For $H \neq \frac{1}{2}$, the fBm B^H has correlated increments :

$$\begin{aligned} \rho(n) &= E(B_1^H(B_{n+1}^H - B_n^H)) \\ &= \frac{1}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right) \\ &\sim H(2H-1)n^{2H-2}, \end{aligned}$$

  as $n \to \infty$.
  (ii) If $H > \frac{1}{2}$, then $\rho(n) > 0$ and $\sum_n \rho(n) = \infty$ (*long memory*).
  (iii) If $H < \frac{1}{2}$, then $\rho(n) < 0$ (*intermittency*) and $\sum_n |\rho(n)| < \infty$.

- $\frac{1}{H}$*-variation :*

  Fix $T > 0$. Set $t_i = \frac{iT}{n}$ for $1 \leq i \leq n$ and define $\Delta B^H_{t_i} = B^H_{t_i} - B^H_{t_{i-1}}$. Then, as $n \to \infty$,

$$\sum_{i=1}^{n} |\Delta B^H_{t_i}|^{\frac{1}{H}} \overset{L^2(\Omega), a.s.}{\longrightarrow} c_H T,$$

  where $c_H = E[|B^H_1|^{\frac{1}{H}}]$.

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$$\sum_{i=1}^{n} |\Delta B^H_{t_i}|^{\frac{1}{H}} \overset{L^2(\Omega),a.s.}{\longrightarrow} c_H T,$$

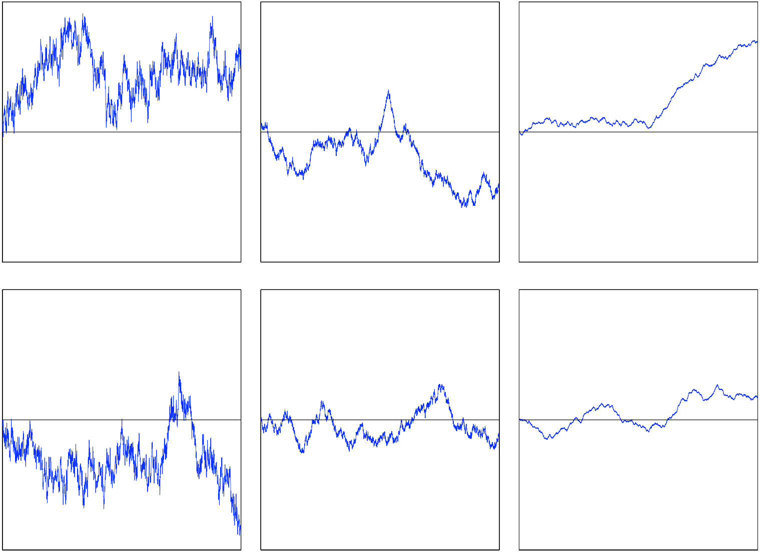
  where $c_H = E[|B^H_1|^{\frac{1}{H}}]$.

  Proof : By the self-similarity, $\sum_{i=1}^n |\Delta B^H_{t_i}|^{\frac{1}{H}}$ has the same law as

$$\frac{T}{n} \sum_{i=1}^{n} |B^H_i - B^H_{i-1}|^{\frac{1}{H}}.$$

  The sequence $\{B^H_i - B^H_{i-1}, i \ge 1\}$ is stationary and ergodic. Therefore, the Ergodic Theorem implies the desired convergence.

Examples of fBm paths :



$H = 0.3$ $H = 0.5$ $H = 0.7$

Fractional noise

- Let $X_k = B_k^H - B_{k-1}^H$. The sequence $\{X_k, k \geq 1\}$ is Gaussian, stationary and centered with covariance

$$\rho(n) = \frac{1}{2}\left(|n+1|^{2H} + |n-1|^{2H} - 2|n|^{2H}\right).$$

We have $\rho(n) \sim H(2H-1)n^{2H-2}$ as $n \to \infty$. Then, for any integer $q \geq 2$ such that $H < 1 - \frac{1}{2q}$, we have

$$\sum_{v \in \mathbb{Z}} |\rho(v)|^q < \infty$$

and the Breuer-Major theorem implies

$$\boxed{\frac{1}{\sqrt{n}} \sum_{k=1}^{n} h_q(B_k^H - B_{k-1}^H) \Rightarrow N(0, \sigma_{H,q}^2),}$$

where $\sigma_{H,q}^2 = q! \sum_{v \in \mathbb{Z}} \rho(v)^q$.

CLT for the q-variation of the fBm

- For a real $q \geq 1$, set $c_q = E[|Z|^q]$, where $Z \sim N(0,1)$.
- The Breuer-Major theorem leads to the following convergence :

Theorem

Suppose $H < \frac{1}{2}$ and q is not an even integer. As $n \to \infty$ we have

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left[n^{qH} |B^H_{\frac{k}{n}} - B^H_{\frac{k-1}{n}}|^q - c_q \right] \Rightarrow N(0, \tilde{\sigma}^2_{H,q}).$$

Proof : Use that $|x|^q - c_q$ has Hermite rank 1.

Rate of convergence for the quadratic variation

- Define for $n \geq 1$,

$$S_n = \sum_{k=1}^{n} (\Delta_{k,n} B^H)^2,$$

where $\Delta_{k,n} B^H = B^H_{\frac{k}{n}} - B^H_{\frac{k-1}{n}}$.

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- Then,

$$n^{2H-1} S_n \stackrel{\text{a.s.}}{\to} 1,$$

n tends to infinity. In fact, by the self-similarity property, $n^{2H-1} S_n$ has the same law as $\frac{1}{n} \sum_{k=1}^{n} (B^H_k - B^H_{k-1})^2$, and the result follows form the Ergodic Theorem.

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- To study the asymptotic normality, consider
$$F_n = \frac{1}{\sigma_n} \sum_{k=1}^{n} \left[n^{2H} (\Delta_{k,n} B^H)^2 - 1 \right] \overset{\mathcal{L}}{=} \frac{1}{\sigma_n} \sum_{k=1}^{n} \left[(B^H_k - B^H_{k-1})^2 - 1 \right],$$
where σ_n is such that $E[F_n^2] = 1$.

Theorem

Assume $H < \frac{3}{4}$. *Then,* $\lim_{n\to\infty} \frac{\sigma^2}{n} = 2\sum_{r\in\mathbb{Z}} \rho^2(r)$ *and*

$$d_{TV}(F_n, Z) \le c_H \times \begin{cases} n^{-\frac{1}{2}} & \text{if } H \in (0, \frac{5}{8}) \\ n^{-\frac{1}{2}}(\log n)^{\frac{3}{2}} & \text{if } H = \frac{5}{8} \\ n^{4\mathbf{H}-3} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}). \end{cases}$$

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- As a consequence,

$$\sqrt{n}(n^{2H-1}S_n - 1) \Rightarrow N\left(0, 2\sum_{r\in\mathbb{Z}} \rho^2(r)\right).$$

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- As a consequence,

$$\sqrt{n}(n^{2H-1} S_n - 1) \Rightarrow N\left(0, 2\sum_{r\in\mathbb{Z}} \rho^2(r)\right).$$

- The estimator of H given by $\widehat{H}_n = \frac{1}{2} - \frac{\log S_n}{2\log n}$ satisfies $\widehat{H}_n \xrightarrow{\text{a.s.}} H$ and

$$\sqrt{n}\log n(\widehat{H}_n - H) \Rightarrow N\left(0, \frac{1}{2}\sum_{r\in\mathbb{Z}} \rho^2(r)\right).$$

Sketch of the proof :

- There exists a sequence $\{e_k, k \geq 1\}$ in $H = L^2([0, T])$ such that

$$\langle e_k, e_j \rangle_H = \rho(k - j).$$

The sequence $\{W(e_k)\}$ has the same law as $\{B_k^H - B_{k-1}^H\}$, and we may replace F_n by

$$G_n = \frac{1}{\sigma_n} \sum_{k=1}^{n} \left[W(e_k)^2 - 1 \right] = I_2(f_n),$$

where $f_n = \frac{1}{\sigma_n} \sum_{k=1}^{n} e_k \otimes e_k$.

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- By the isometry property of multiple integrals,

$$1 = E[G_n^2] = 2\|f_n\|^2_{L^2([0,T]^2)} = \frac{2}{\sigma_n^2} \sum_{k,j=1}^n \rho^2(k-j) = \frac{2n}{\sigma_n^2} \sum_{|r|<n} \left(1 - \frac{|r|}{n}\right) \rho^2(r).$$

Since $\sum_r \rho^2(r) < \infty$, because $H < \frac{3}{4}$, we deduce that $\lim_{n\to\infty} \frac{\sigma^2}{n} = 2\sum_{r\in\mathbb{Z}} \rho^2(r)$.

- We can write $D_r[I_2(f_n)] = 2I_1(f_n(\cdot, r))$ and

$$\|D[I_2(f_n)]\|_H^2 = 4\left(I_2(f_n \otimes_1 f_n) + \|f_n\|_H^2\right) = 4I_2(f_n \otimes_1 f_n) + 2.$$

Therefore,

$$\begin{aligned}
\mathrm{Var}\left(\|D[I_2(f_n)]\|_H^2\right) &= 16E\left[(I_2(f_n \otimes_1 f_n))^2\right] \\
&= 8\|f_n \otimes_1 f_n\|_{L^2([0,T]^2)}^2 \\
&= \frac{16}{\sigma_n^4} \sum_{k,j,i,\ell=1}^{n} \rho(k-j)\rho(i-\ell)\rho(k-i)\rho(j-\ell) \\
&\le \frac{16}{\sigma_n^4} \sum_{i,\ell=1}^{n} (\rho_n * \rho_n)(i-\ell)^2 \\
&\le \frac{16n}{\sigma_n^4} \sum_{k\in\mathbb{Z}} (\rho_n * \rho_n)(k)^2 = \frac{16n}{\sigma_n^4} \|\rho_n * \rho_n\|_{\ell^2(\mathbb{Z})}^2,
\end{aligned}$$

where $\rho_n(k) = |\rho(k)|\mathbf{1}_{\{|k|\le n-1\}}$.

- Applying Young's inequality yields

$$\|\rho_n * \rho_n\|^2_{\ell^2(\mathbb{Z})} \le \|\rho_n\|^4_{\ell^{4/3}(\mathbb{Z})},$$

so that

$$\mathrm{Var}\left(\|D[I_2(f_n)]\|^2_H\right) \le \frac{16n}{\sigma_n^4}\left(\sum_{|k|<n} |\rho(k)|^{\frac{4}{3}}\right)^3 .$$

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- Thus,

$$d_{TV}(F_n, Z) \leq \frac{4\sqrt{n}}{\sigma_n^2}\left(\sum_{|k|<n} |\rho(k)|^{\frac{4}{3}}\right)^{\frac{3}{2}}$$

and the result follows from $\rho(k) \sim H(2H-1)|k|^{2H-2}$ as $|k| \to \infty$.

Remark :

- Nourdin-Peccati '13 proved the following optimal version of the fourth moment theorem (assuming $E[F_n^2] = 1$) :

$$c\mathbf{M}(F_n) \le d_{TV}(F_n, Z) \le C\mathbf{M}(F_n),$$

where $\mathbf{M}(F_n) = \max(|E[F_n^3]|, E[F_n^4] - 3)$.

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- As a consequence, the sequence $F_n = \frac{1}{\sigma_n} \sum_{k=1}^n \left[(B_k^H - B_{k-1}^H)^2 - 1\right]$ satisfies :

$$d_{TV}(F_n, Z) \le c_H \times \begin{cases} n^{-\frac{1}{2}} & \text{if } H \in (0, \frac{2}{3}) \\ n^{-\frac{1}{2}}(\log n)^2 & \text{if } H = \frac{2}{3} \\ n^{6H-\frac{9}{2}} & \text{if } H \in (\frac{2}{3}, \frac{3}{4}) \end{cases}$$

Convergence of densities

- The total variation distance is equivalent to the L^1-norm of the densities :

$$d_{TV}(F, Z) = \int_{\mathbb{R}} |p_F(x) - \phi(x)| dx,$$

where $Z \sim N(0, 1)$ and ϕ is the density of Z.

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- Uniform convergence, however, requires stronger hypotheses.

Theorem (Hu-Lu-N. '13)

Let $F \in \mathcal{H}_q$, $q \geq 2$, be such that $E(F^2) = 1$ and $E(\|DF\|_H^{-6}) \leq M$. Then,

$$\sup_{s \in \mathbb{R}} |p_F(x) - \phi(x)| \leq C_{M,q} \sqrt{E(F^4) - 3}.$$

Sketch of the proof :
(i) Density formula :

$$p_F(x) = E\left[\mathbf{1}_{\{F>x\}}\delta\left(\frac{DF}{\|DF\|_H^2}\right)\right].$$

It follows from :

$$E(\phi'(F)) = E\left(\frac{\langle D\phi(F), DF\rangle_H}{\|DF\|_H^2}\right) = E\left(\phi(F)\delta\left(\frac{DF}{\|DF\|_H^2}\right)\right).$$

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(ii) Using $\delta(Gu) = G\delta(u) - \langle DG, u\rangle_H$, and $\delta DF = qF$, we can write

$$\begin{aligned}
p_F(x) &= E\left[\mathbf{1}_{\{F>x\}}\delta\left(\frac{DF}{\|DF\|_H^2}\right)\right] \\
&= E\left[\mathbf{1}_{\{F>x\}}\frac{qF}{\|DF\|_H^2}\right] - E[\mathbf{1}_{\{F>x\}}\langle DF, D(\|DF\|_H^{-2})\rangle_H] \\
&= E[\mathbf{1}_{\{F>x\}}F] + E[F(q\|DF\|_H^{-2} - 1)] - E[\mathbf{1}_{\{F>x\}}\langle DF, D(\|DF\|_H^{-2})\rangle_H] \\
&= E[\mathbf{1}_{\{F>x\}}F] + R_n.
\end{aligned}$$

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(iv) Taking into account that

$$\phi(x) = E[\mathbf{1}_{\{Z>x\}} Z],$$

where $Z \sim N(0,1)$, it suffices to estimate the difference

$$E[\mathbf{1}_{\{F>x\}} F] - E[\mathbf{1}_{\{Z>x\}} Z],$$

which can be done by Stein's method and Malliavin calculus.

Example 1

- Let $q = 2$ and
$$F = \sum_{i=1}^{\infty} \lambda_i (X(e_i)^2 - 1),$$
where $\{e_i, i \geq 1\}$ is a complete orthonormal system in H and λ_i is a decreasing sequence of positive numbers such that $\sum_{i=1}^{\infty} \lambda_i^2 < \infty$.

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- Suppose $E[F^2] = 1$. Then, if $\lambda_N \neq 0$ for some $N > 4$, we obtain
$$\sup_{x \in \mathbb{R}} |p_F(x) - \phi(x)| \leq C_{N,\lambda_N} \sqrt{\sum_{i=1}^{\infty} \lambda_i^4}.$$

Example 2 (Brauer-Major theorem revisited)

Fix $q \geq 2$ and consider the sequence

$$V_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \sum_{j=d}^{q} a_j h_j(X_k), \quad a_d \neq 0,$$

where $X = \{X_k, \, k \in \mathbb{Z}\}$ is a centered Gaussian stationary sequence with unit variance and covariance $\rho(v)$.

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Theorem (Hu-N.-Tindel-Xu '14)

Suppose the spectral density of X, f_ρ, satisfies $\log(f_\rho) \in L^1([-\pi, \pi])$. Assume $\sum_{v\in\mathbb{Z}} |\rho(v)|^d < \infty$. Set $\sigma^2 := \sum_{j=d}^{q} j! a_j^2 \sum_{v\in\mathbb{Z}} \rho(v)^j \in (0, \infty)$. Then for any $p \geq 1$, there exists n_0 such that

$$\sup_{n\geq n_0} E[\|DV_n\|_H^{-p}] < \infty. \tag{1}$$

Therefore, if $q = d$ and $F_n = V_n / \sqrt{E[V_n^2]}$, we have

$$\sup_{x\in\mathbb{R}} |p_{F_n}(x) - \phi(x)| \leq c\sqrt{E[F_n^4] - 3}.$$

Sketch of the proof :

- From the non-causal representation $X_k = \sum_{j=0}^{\infty} \psi_j w_{k-j}$, where $\{w_k, k \in \mathbb{Z}\}$ is a discrete Gaussian white noise, it follows that

$$\|DV_n\|_H^2 \geq \frac{1}{n} \sum_{m=1}^{n} \left(\sum_{k=m}^{n} \sum_{j=d}^{q} a_j h_j'(X_k) \psi_{k-m} \right)^2 := G_n.$$

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- We use the estimate

$$G_n^{-\frac{p}{2}} \leq \prod_{i=1}^{N} (G_n^i)^{-\frac{p}{2N}}$$

and we apply the Carbery-Wright inequality to control the expectation of $(G_n^i)^{-\frac{p}{2N}}$ if $\frac{p}{2N}$ is small enough.

Lemma (Carbery-Wright inequality, '01)

There is a universal constant $c > 0$ such that, for any polynomial $Q : \mathbb{R}^n \to \mathbb{R}$ of degree at most d and any $\alpha > 0$, we have

$$E[Q(X_1,\dots,X_n)^2]^{\frac{1}{2d}} P(|Q(X_1,\dots,X_n)| \le \alpha) \le cd\alpha^{\frac{1}{d}},$$

where $X_1,\dots,X_n$ are independent random variables with law $N(0,1)$.

Particular case :

- Let $\{X_k = B^H_k - B^H_{k-1}\,, k \geq 1\}$. The spectral density satisfies $\log(f_{\rho_H}) \in L^1([-\pi, \pi])$.

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- Let $\{X_k = B^H_k - B^H_{k-1}\,, k \geq 1\}$. The spectral density satisfies $\log(f_{\rho_H}) \in L^1([-\pi, \pi])$.
- As a consequence, we obtain the uniform convergence of densities to ϕ for the sequence of Hermite variations $F_n = V_n/E[V_n^2]$, where

$$V_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} h_q(n^H \Delta_{k,n} B^H), \quad q \geq 2,$$

for $0 < H < 1 - \frac{1}{2q}$, where $\Delta_{k,n} B^H = B^H_{k/n} - B^H_{(k-1)/n}$.

Particular case :

- Let $\{X_k = B^H_k - B^H_{k-1}\,, k \geq 1\}$. The spectral density satisfies $\log(f_{\rho_H}) \in L^1([-\pi,\pi])$.
- As a consequence, we obtain the uniform convergence of densities to ϕ for the sequence of Hermite variations $F_n = V_n/E[V_n^2]$, where

$$V_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} h_q(n^H \Delta_{k,n} B^H), \quad q \geq 2,$$

  for $0 < H < 1 - \frac{1}{2q}$, where $\Delta_{k,n} B^H = B^H_{k/n} - B^H_{(k-1)/n}$.
- If $H \in (0, \frac{3}{4})$ and $q = 2$, we have

$$\sup_{x\in\mathbb{R}} |p_{F_n}(x) - \phi(x)| \leq c\sqrt{E(F_n^4) - 3} \leq c_H \begin{cases} n^{-\frac{1}{2}} & \text{if } H \in (0, \frac{5}{8}) \\ n^{-\frac{1}{2}} (\log n)^{\frac{3}{2}} & \text{if } H = \frac{5}{8} \\ n^{4H-3} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}) \end{cases}$$

Generalizations :

(i) One can show the uniform approximation of the mth derivative of p_F by the corresponding mth derivative of the Gaussian density $\phi^{(m)}$ under the stronger assumption $E(\|DF\|_H^{-\beta}) < \infty$ for some $\beta > 6m + 6\left(\lfloor \frac{m}{2} \rfloor \vee 1\right)$.

Generalizations :

(i) One can show the uniform approximation of the mth derivative of p_F by the corresponding mth derivative of the Gaussian density $\phi^{(m)}$ under the stronger assumption $E(\|DF\|_H^{-\beta}) < \infty$ for some $\beta > 6m + 6\left(\lfloor \frac{m}{2} \rfloor \vee 1\right)$.

(ii) Consider a d-dimensional vector F, whose components are in a fixed chaos, and such that $E[(\det \gamma_F)^{-p}] < \infty$ for all p, where γ_F denotes the Malliavin matrix of F. In this case for any multi-index $\beta = (\beta_1, \ldots, \beta_k)$, $1 \leq \beta_i \leq d$, one can show

$$\sup_{x \in \mathbb{R}^d} |\partial_\beta f_F(x) - \partial_\beta \phi_d(x)| \leq c\left(|C - I|^{\frac{1}{2}} + \sum_{j=1}^{d} \sqrt{E[F_j^4] - 3(E[F_j^2])^2}\right)$$

where C is the covariance matrix of F, ϕ_d is the standard d-dimensional normal density, and $\partial_\beta = \frac{\partial^k}{\partial x_{\beta_1} \cdots \partial x_{\beta_k}}$.

Non-Central Limit Theorem for multiple Skorohod integrals

Theorem (Nourdin-N. '10)

Fix an integer $q \geq 1$. Let $F_n = \delta^q(u_n)$, where u_n is a symmetric element in $\mathbb{D}^{2q,2q}(H^{\otimes q})$. Suppose that $\sup_n \|F_n\|_{q,p} < \infty$ for any $p \geq 2$ and

(i) $\langle u_n, h \rangle_{H^{\otimes q}} \to 0$ *in* L^1 *for any* $h \in H^{\otimes q}$, *and* $u_n \otimes_j D^j F_n \to 0$ *in* L^2 *for all* $j = 1, \dots, q-1$,

(ii) $\langle u_n, D^q F_n \rangle_{H^{\otimes q}} \xrightarrow{L^1} S^2$.

Then, F_n *converges stably in law to* $N(0, S^2)$.

- The sequence F_n converges stably to $N(0, S^2)$ if for any bounded random variable Z measurable with respect to W and for any $\lambda \in \mathbb{R}$

$$\lim_{n\to\infty} E\left(e^{i\lambda F_n} Z\right) = E\left(e^{-\frac{\lambda^2}{2} S^2} Z\right).$$

Sketch of the proof

- To simplify, assume $q = 1$.
- Fix $\Phi_m = (W(e_1), \dots, W(e_m))$, where e_i is a basis of H. It suffices to show that for each m

$$(F_n, \Phi_m) \stackrel{\text{Law}}{\to} (F_\infty, \Phi_m),$$

where

$$E\left(e^{i\lambda F_\infty} | W\right) = e^{-\frac{\lambda^2}{2} S^2}. \tag{2}$$

- Because the laws of (F_n, Φ) are tight, we can assume the above convergence in law, and it suffices to show that the limit satisfies (2)
- Set $Y = g(\Phi_m)$, where $g \in C_b^\infty(\mathbb{R}^m)$ and define

$$\phi_n(\lambda) = E(e^{i\lambda F_n} Y).$$

- We compute the limit of $\phi_n'(\lambda)$ in two ways :
  1. Using weak convergence :
  $$\phi_n'(\lambda) = iE(e^{i\lambda F_n} F_n Y) \to iE(e^{i\lambda F_\infty} F_\infty Y).$$
  2. Using Malliavin calculus and our assumptions :
  $$\begin{aligned} \phi_n'(\lambda) &= iE(e^{i\lambda F_n} F_n Y) = iE(e^{i\lambda F_n} \delta(u_n) Y) \\ &= iE\left(\left\langle D\left(e^{i\lambda F_n} Y\right), u_n\right\rangle_H\right) \\ &= -\lambda E\left(e^{i\lambda F_n} \langle u_n, DF_n\rangle_H Y\right) + iE\left(e^{i\lambda F_n}\langle u_n, DY\rangle_H\right) \\ &\quad \to -\lambda E(e^{i\lambda F_\infty} S^2 Y) \end{aligned}$$
- As a consequence,
$$iE(e^{i\lambda F_\infty} F_\infty Y) = -\lambda E(e^{i\lambda F_\infty} S^2 Y).$$

- This leads to a linear differential equation satisfied by the conditional characteristic function of F_∞ :

$$\frac{\partial}{\partial\lambda} E(e^{i\lambda F_\infty}|W) = -S^2 \lambda E(e^{i\lambda F_\infty}|W),$$

and we obtain

$$E(e^{i\lambda F_\infty}|W) = e^{-\frac{\lambda^2}{2}S^2}.$$

Corollary

A sequence $F_n = \delta(u_n)$*, where* $u_n \in \mathbb{D}^{2,2}(H)$*, such that* $\sup_n \|F_n\|_{2,p} < \infty$ *for any* $p \geq 1$*, converges stably to* $N(0, S^2)$ *if :*

(i) $\langle u_n, h \rangle_H \to 0$ *in* $L^1(\Omega)$*, for every* $h \in H$*.*

(ii) $\langle u_n, DF_n \rangle_H \to S^2$ *in* $L^1(\Omega)$*.*

- Notice that

$$\langle u_n, DF_n \rangle_H = \|u_n\|_H^2 + \langle u_n, \delta(Du_n) \rangle_H.$$

Therefore, a sufficient condition for (ii) is :

(ii') $\|u_n\|_H^2 \xrightarrow{L^1} S^2$ and $\langle u_n, \delta(Du_n) \rangle_H \xrightarrow{L^1} 0$.

- Comparison with the *Asymptotic Knight Theorem* for Brownian martingales (Revuz-Yor) :

  If $\{u_n, n \geq 1\}$ are square-integrable adapted processes, then, $F_n = \delta(u_n) = \int_0^\infty u_n(s) dW_s$ and the stable convergence of F_n to $N(0, S^2)$ is implied by the following conditions :

  (A) $\int_0^t u_n(s) ds \xrightarrow{P} 0$, uniformly in t in compact sets.
  (B) $\int_0^\infty u_n(s)^2 ds \to S^2$ in $L^1(\Omega)$.

Applications :

- Weighted variations of the fractional Brownian motion (Nourdin, Réveillac, Tudor, N.) : Assume $\frac{1}{2q} < H < 1 - \frac{1}{2q}$. Then,

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(B^H_{\frac{k-1}{n}}) h_q(n^H \Delta B^H_{k,n}) \xrightarrow{\text{Stably}} \sigma_{H,q} \int_0^1 f(B^H_s) dW_s,$$

  where W is a Brownian motion independent of B^H.

- Non-central limit theorem for symmetric integrals with respect to the fractional Brownian motion for critical values of the Hurst parameter (Burdzy, Swanson, Nourdin, Réveillac, Harnett, N.)

Rate of convergence

Theorem (Nourdin-Peccati-N. '16)

Let $F = \delta(u)$, where $u \in \mathbb{D}^{1,2}(H)$ and $F \in \mathbb{D}^{1,2}$. Let $S \geq 0$ be such that $S^2 \in \mathbb{D}^{1,2}$ and let η be a $N(0,1)$ random variable independent of X. Assume that $\varphi \in C_b^3$. Then :

$$|E[\varphi(F] - E[\varphi(S\eta)]| \leq \frac{1}{2}\|\varphi''\|_\infty E\left[|\langle u, DF\rangle_H - S^2|\right] + \frac{1}{3}\|\varphi'''\|_\infty E\left[|\langle u, DS^2\rangle_H|\right].$$

Proof :

- Fix $\epsilon > 0$ and set $S_\epsilon = \sqrt{S^2 + \epsilon} \in \mathbb{D}^{1,2}$.

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- Let $g(t) = E[\varphi(\sqrt{t}F + \sqrt{1-t}S_\epsilon \eta)]$, $t \in [0,1]$. Then,

$$E[\varphi(F)] - E[\varphi(S_\epsilon \eta)] = g(1) - g(0) = \int_0^1 g'(t)dt.$$

Proof :

- Fix $\epsilon > 0$ and set $S_\epsilon = \sqrt{S^2 + \epsilon} \in \mathbb{D}^{1,2}$.
- Let $g(t) = E[\varphi(\sqrt{t}F + \sqrt{1-t}S_\epsilon \eta)]$, $t \in [0,1]$. Then,

$$E[\varphi(F)] - E[\varphi(S_\epsilon \eta)] = g(1) - g(0) = \int_0^1 g'(t)dt.$$

- Integrating by parts using $F = \delta(u)$, yields

$$\begin{aligned} g'(t) &= \frac{1}{2} E\left[\varphi'(\sqrt{t}F + \sqrt{1-t}S_\epsilon \eta)\left(\frac{F}{\sqrt{t}} - \frac{S_\epsilon \eta}{\sqrt{1-t}}\right)\right] \\ &= \frac{1}{2} E\Bigg[\varphi''(\sqrt{t}F + \sqrt{1-t}S_\epsilon \eta) \\ &\qquad \times \left(\langle u, DF\rangle_H + \frac{\sqrt{1-t}}{\sqrt{t}}\eta\langle u, DS_\epsilon\rangle_H - S_\epsilon^2\right)\Bigg]. \end{aligned}$$

- Integrating again by parts with respect to the law of η yields

$$
\begin{aligned}
g'(t) &= \frac{1}{2} E\left[\varphi''(\sqrt{t}F + \sqrt{1-t}S_\epsilon \eta)\left(\langle u, DF\rangle_{\mathfrak{H}} - S_\epsilon^2\right)\right] \\
&\quad + \frac{1-t}{4\sqrt{t}} E\left[\varphi'''(\sqrt{t}F + \sqrt{1-t}S_\epsilon \eta)\langle u, DS^2\rangle_H\right],
\end{aligned}
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where we have used the fact that $S_\epsilon DS_\epsilon = \frac{1}{2} DS_\epsilon^2 = \frac{1}{2} DS^2$.

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where we have used the fact that $S_\epsilon DS_\epsilon = \frac{1}{2}DS_\epsilon^2 = \frac{1}{2}DS^2$.

- Finally, integrating in t,

$$
\begin{aligned}
|E[\varphi(F)] - E[\varphi(S_\epsilon\eta)]| &\le \frac{1}{2}\|\varphi''\|_\infty E\left[|\langle u, DF\rangle_H - S^2 - \epsilon|\right] \\
&\quad + \|\varphi'''\|_\infty E\left[|\langle u, DS^2\rangle_H|\right] \int_0^1 \frac{1-t}{4\sqrt{t}} dt,
\end{aligned}
$$

and the conclusion follows because $\int_0^1 \frac{1-t}{4\sqrt{t}} dt = \frac{1}{3}$.

Example (Weighted quadratic variation of the fBm) :

$$F_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(B^H_{(k-1)/n}) \left[(n^H \Delta_{k,n} B^H)^2 - 1 \right].$$

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Theorem (Nourdin-N.-Peccati '16)

Let $H \in (\frac{1}{4}, \frac{3}{4})$ and $f \in C^4(\mathbb{R})$ such that $|f^{(i)}(x)| \le c_1 \exp(c_2|x|^\beta)$ for some $\beta < 2$ and for $i = 0, \dots, 4$. Let

$$S = \sqrt{\sigma_H \int_0^1 f^2(B^H_s) ds},$$

with $\sigma_H^2 = \sum_{k=-\infty}^{\infty} \rho_H(k)^2$. Suppose $E[S^{-\alpha}] < \infty$ for some $\alpha > 2$. Then,

$$|E[\varphi(F_n)] - E[\varphi(S\eta)]| \le C_{f,H} \max_{1 \le i \le 5} \|\varphi^{(i)}\|_\infty \; n^{-(|2H-\frac{1}{2}| \wedge |2H-\frac{3}{2}|)},$$

where η is a $N(0,1)$ random variable independent of B^H.